

IMPLICIT STRUCTURE IN 2-REPRESENTATIONS OF QUANTUM GROUPS

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ABSTRACT. Given a strong 2-representation of a Kac-Moody Lie algebra (in the sense of Rouquier) we show how to extend it to a 2-representation of categorified quantum groups (in the sense of Khovanov-Lauda). This involves checking certain extra 2-relations which are explicit in the definition by Khovanov-Lauda and, as it turns out, implicit in Rouquier's definition. Some applications are also discussed.

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1. INTRODUCTION

Higher representation theory arises in the study of actions by Lie algebras, such as \mathfrak{sl}_2 , on categories rather than vector spaces. Instead of linear maps between vector spaces, such as E and F , there are functors \mathbf{E} and \mathbf{F} between categories. Subsequently, the work of Chuang and Rouquier [9] revealed the importance of studying natural transformations between these functors which are invisible in the usual representation theory

These natural transformations also satisfy certain relations. All of this data, namely the functors and the natural transformations, is captured and formalized in the concept of a categorified quantum group. These are 2-categories which were introduced and studied by Khovanov-Lauda in [17, 12, 14, 13]. Closely related 2-categories were independently introduced by Rouquier in [22]. In these 2-categories, the functors correspond to 1-morphisms while the natural transformations correspond to 2-morphisms.

One obvious difference between the approaches of Khovanov-Lauda and Rouquier is certain extra 2-relations which occur in [12, 14, 13] but not in [22]. These additional relations induce explicit 2-isomorphisms lifting certain relations of the quantum group, and allow one to directly relate the split Grothendieck ring of these 2-categories with the integral version of the associated quantum group. The Khovanov-Lauda approach uses a diagrammatic framework where 2-morphisms are depicted diagrammatically using string-like diagrams (as explained later). When interpreted diagrammatically, the extra 2-relations in the Khovanov-Lauda approach tell you how to simplify an arbitrary string diagram (depicting a 2-morphism).

The main result of this paper (Theorem 1.1) shows that these extra relations are essentially forced upon us. By adding the requirement that there are no negative degree endomorphisms of identity 1-morphisms we show that a representation of 2-Kac-Moody algebras in the sense of Rouquier extends to a 2-representation of categorified quantum groups in the sense of Khovanov-Lauda. In the process of proving this we also clarify the definition of these categorified quantum groups by defining a 2-category $\mathcal{U}_Q(\mathfrak{g})$ for arbitrary Kac-Moody Lie algebras (and arbitrary KLR algebras). These definitions differ slightly from those currently in the literature but, as mentioned above, are forced upon us.

We also show that certain biadjointness between functors \mathbf{E} and \mathbf{F} holds for free (Proposition 3.2). This means that it suffices to know that the functor \mathbf{E} has left and right adjoints. The fact that the left and right adjoints are isomorphic (up to a grading shift) is then a formal consequence. This fact is also noticed by Rouquier in [22].

We finish by briefly discussing some applications (Section 7). In particular, it was conjectured in [13] that the 2-category $\mathcal{U}_Q(\mathfrak{g})$ should admit various 2-representations. Theorem 1.1 resolves one of these conjectures by allowing us to deduce the existence of a 2-representation of the 2-category $\mathcal{U}_Q(\mathfrak{g})$ on derived categories of coherent sheaves on cotangent bundles to Grassmannians. While the added condition that there are no negative degree endomorphisms of the identity 1-morphisms is a natural condition to verify in most geometric settings, checking this condition in algebraic settings such as categories of projective modules over cyclotomic quotients of KLR algebras can be more challenging.

In the remaining part of the introduction we sketch the definition of the 2-category $\mathcal{U}_Q(\mathfrak{g})$ and what it means to have a Q -strong 2-representation of \mathfrak{g} . A Q -strong 2-representation of \mathfrak{g} is essentially a 2-Kac-Moody representation in the sense of Rouquier with one additional condition that there are no negative degree endomorphisms of the identity 1-morphisms 1_λ . Our main result (Theorem 1.1) states that a Q -strong 2-representation of \mathfrak{g} induces a 2-representation of $\mathcal{U}_Q(\mathfrak{g})$.

This article provides a comparison between the 2-representation theories of the 2-categories introduced by Rouquier and that of the 2-category $\mathcal{U}_Q(\mathfrak{g})$. In practice, one checks the Q -strong 2-representation conditions, which are simpler and easier to verify, and uses Theorem 1.1 to define an action of the 2-category $\mathcal{U}_Q(\mathfrak{g})$. The 2-category $\mathcal{U}_Q(\mathfrak{g})$ is more elaborate and includes more relations telling you how to simplify any string diagram.

In [22] the notion of a 2-Kac-Moody representation is formalized by introducing several 2-categories associated to Kac-Moody algebras. Isomorphisms lifting quantum Serre relations are obtained via a localization procedure on certain 1-morphisms in these 2-categories. A 2-Kac-Moody representation is then defined as a 2-functor from one of these 2-categories to an appropriate 2-category \mathcal{K} . The results in this article are insufficient to deduce that any version of the 2-categories Rouquier defines has Grothendieck ring isomorphic to the quantum group. This is because we do not know how to verify the condition that there are no negative degree endomorphisms of $\mathbf{1}_\lambda$ (which is required for Theorem 1.1 to hold) from the localization procedure used in Rouquier's approach.

1.1. The 2-category $\mathcal{U}_Q(\mathfrak{g})$. To a Cartan datum and a choice of scalars Q (see section 2.1.1) we define a 2-category $\mathcal{U}_Q(\mathfrak{g})$. Note that the 2-category $\mathcal{U}(\mathfrak{g})$ from [13] corresponds to the choice of scalars with $t_{ij} = t_{ji} = 1$ and $s_{ij}^{pq} = 0$ for all $i, j \in I$.

Definition 1.1. The 2-category $\mathcal{U}_Q(\mathfrak{g})$ is the graded additive \mathbb{k} -linear 2-category (see section 2.1.2) consisting of:

- objects λ for $\lambda \in X$.
- 1-morphisms are formal direct sums of compositions of

$$\mathbf{1}_\lambda, \quad \mathbf{1}_{\lambda+\alpha_i} \mathcal{E}_i = \mathbf{1}_{\lambda+\alpha_i} \mathcal{E}_i \mathbf{1}_\lambda = \mathcal{E}_i \mathbf{1}_\lambda \quad \text{and} \quad \mathbf{1}_{\lambda-\alpha_i} \mathcal{F}_i = \mathbf{1}_{\lambda-\alpha_i} \mathcal{F}_i \mathbf{1}_\lambda = \mathcal{F}_i \mathbf{1}_\lambda.$$

for $i \in I$ and $\lambda \in X$.

- 2-morphisms are \mathbb{k} -vector spaces spanned by compositions of (decorated) tangle-like diagrams illustrated below.

$$\begin{array}{ll} \begin{array}{c} \lambda + \alpha_i \\ \downarrow \\ \bullet \\ \downarrow \\ i \end{array} \quad \lambda : \mathcal{E}_i \mathbf{1}_\lambda \rightarrow \mathcal{E}_i \mathbf{1}_\lambda \langle (\alpha_i, \alpha_i) \rangle & \begin{array}{c} \lambda - \alpha_i \\ \downarrow \\ \bullet \\ \downarrow \\ i \end{array} \quad \lambda : \mathcal{F}_i \mathbf{1}_\lambda \rightarrow \mathcal{F}_i \mathbf{1}_\lambda \langle (\alpha_i, \alpha_i) \rangle \\ \\ \begin{array}{c} \nearrow \quad \searrow \\ i \quad j \end{array} \quad \lambda : \mathcal{E}_i \mathcal{E}_j \mathbf{1}_\lambda \rightarrow \mathcal{E}_j \mathcal{E}_i \mathbf{1}_\lambda \langle (\alpha_i, \alpha_j) \rangle & \begin{array}{c} \nwarrow \quad \swarrow \\ i \quad j \end{array} \quad \lambda : \mathcal{F}_i \mathcal{F}_j \mathbf{1}_\lambda \rightarrow \mathcal{F}_j \mathcal{F}_i \mathbf{1}_\lambda \langle (\alpha_i, \alpha_j) \rangle \\ \\ \begin{array}{c} \cup \\ i \end{array} \quad \lambda : \mathbf{1}_\lambda \rightarrow \mathcal{F}_i \mathcal{E}_i \mathbf{1}_\lambda \langle d_i + (\lambda, \alpha_i) \rangle & \begin{array}{c} \cap \\ i \end{array} \quad \lambda : \mathbf{1}_\lambda \rightarrow \mathcal{E}_i \mathcal{F}_i \mathbf{1}_\lambda \langle d_i - (\lambda, \alpha_i) \rangle \\ \\ \begin{array}{c} \curvearrowright \\ i \end{array} \quad \lambda : \mathcal{F}_i \mathcal{E}_i \mathbf{1}_\lambda \rightarrow \mathbf{1}_\lambda \langle d_i + (\lambda, \alpha_i) \rangle & \begin{array}{c} \curvearrowleft \\ i \end{array} \quad \lambda : \mathcal{E}_i \mathcal{F}_i \mathbf{1}_\lambda \rightarrow \mathbf{1}_\lambda \langle d_i - (\lambda, \alpha_i) \rangle \end{array}$$

Here we follow the conventions in [17] and read diagrams from right to left and bottom to top. The identity 2-morphism of the 1-morphism $\mathcal{E}_i \mathbf{1}_\lambda$ is represented by an upward oriented line labeled by i (likewise, the identity 2-morphism of $\mathcal{F}_i \mathbf{1}_\lambda$ is represented by a downward oriented line labeled i).

The 2-morphisms satisfy the following relations:

- (1) The 1-morphisms $\mathcal{E}_i \mathbf{1}_\lambda$ and $\mathcal{F}_i \mathbf{1}_\lambda$ are biadjoint (up to a specified degree shift). Moreover, the 2-morphisms are Q -cyclic with respect to this biadjoint structure (see section 2.2).

- (2) The \mathcal{E} 's carry an action of the KLR algebra¹ associated to Q , while the \mathcal{F} 's carry an action of the KLR algebra associated to Q' where Q' is determined from Q by a simple procedure (see sections 2.1.1 and 2.3).
- (3) When $i \neq j$ one has the mixed relations (see section 2.4) relating $\mathcal{E}_i \mathcal{F}_j$ and $\mathcal{F}_j \mathcal{E}_i$.
- (4) Dotted bubbles of negative degree are zero while dotted bubbles of degree zero are equal to the identity (see section 2.5).
- (5) For any $i \in I$ one has the extended \mathfrak{sl}_2 relations (see section 2.6).

The 2-category $\dot{\mathcal{U}}_Q(\mathfrak{g})$ is the idempotent completion of the 2-category $\mathcal{U}_Q(\mathfrak{g})$.

Remark. Note that the notion of Q -cyclic biadjointness, and the precise form of the mixed relations in the 2-category $\mathcal{U}_Q(\mathfrak{g})$ have not appeared in the literature before.

1.2. A Q -strong 2-representation of \mathfrak{g} .

Definition 1.2. A Q -strong 2-representation of \mathfrak{g} consists of a graded, additive \mathbb{k} -linear idempotent complete 2-category \mathcal{K} where:

- The objects of \mathcal{K} are indexed by the weights $\lambda \in X$.
- There are identity 1-morphisms $\mathbb{1}_\lambda$ for each λ , as well as 1-morphisms $E_i \mathbb{1}_\lambda : \lambda \rightarrow \lambda + \alpha_i$ in \mathcal{K} . We also assume that $E_i \mathbb{1}_\lambda$ has both right and left adjoints and define the 1-morphism $\mathbb{1}_\lambda F_i : \lambda + \alpha_i \rightarrow \lambda$ as

$$(1.1) \quad \mathbb{1}_\lambda F_i := (E_i \mathbb{1}_\lambda)_R \langle -(\alpha_i, \lambda) - d_i \rangle,$$

where $d_i = (\alpha_i, \alpha_i)/2$. Other 1-morphisms are obtained from these by taking composites, direct sums, grading shifts, and adding images of idempotent 2-morphisms. See section 2.1.1 for more details on the notation above.

On this data we impose the following conditions:

- (1) (Integrability) The object $\lambda + r\alpha_i$ is isomorphic to the zero object for $r \gg 0$ or $r \ll 0$.
- (2) $\text{Hom}_{\mathcal{K}}(\mathbb{1}_\lambda, \mathbb{1}_\lambda \langle l \rangle)$ is zero if $l < 0$ and one-dimensional if $l = 0$. Moreover, the space of 2-morphisms between any two 1-morphisms is finite dimensional.
- (3) We have the following isomorphisms in \mathcal{K} :

$$F_i E_i \mathbb{1}_\lambda \cong E_i F_i \mathbb{1}_\lambda \oplus_{[-\langle i, \lambda \rangle]_i} \mathbb{1}_\lambda \text{ if } \langle i, \lambda \rangle \leq 0$$

$$E_i F_i \mathbb{1}_\lambda \cong F_i E_i \mathbb{1}_\lambda \oplus_{[\langle i, \lambda \rangle]_i} \mathbb{1}_\lambda \text{ if } \langle i, \lambda \rangle \geq 0.$$

- (4) The E 's carry an action of the KLR algebra associated to Q .
- (5) If $i \neq j \in I$ then $F_j E_i \mathbb{1}_\lambda \cong E_i F_j \mathbb{1}_\lambda$ in \mathcal{K} .

When Q is clear from the context we will simply call this a strong 2-representation of \mathfrak{g} . Note that we do not require an action of the KLR algebra on the F 's as the existence of such an action will follow formally from the action on the E 's.

Important convention. The integrability condition above implies that “most” objects are isomorphic to the zero object. If λ is the zero object then, by definition, $\text{Hom}_{\mathcal{K}}(\mathbb{1}_\lambda, \mathbb{1}_\lambda \langle l \rangle) = 0$ for all l . So, to be precise, condition (2) above should say that $\text{Hom}_{\mathcal{K}}(\mathbb{1}_\lambda, \mathbb{1}_\lambda)$ is one-dimensional if λ is non-zero. There are many other such instances later in this paper. The convention is that any statement about a certain Hom being non-zero (for example, the claims in lemma 3.3 or corollaries 3.4 and 3.5) assumes that all weights involved are non-zero (otherwise the Hom space is automatically zero).

Remark.

¹The KLR algebra is also called the quiver Hecke algebra in the literature.

- (1) Our definition of a Q -strong 2-representation of \mathfrak{g} is nearly identical to Rouquier's definition of a 2-Kac-Moody representation. The exception is condition (2) which is used repeatedly later in this article.
- (2) The integrability condition is used to show that the functors E_s and F_s are biadjoint and that the 1-morphisms E_i have no negative degree endomorphisms. So if one adds these conditions to the definition then one can drop the integrability condition. Usually this is a poor compromise since checking biadjointness can be difficult. However, there are situations where checking biadjointness and the vanishing of negative degree 2-morphisms is easy but where the integrability condition fails.
- (3) The condition about finite dimensional Homs is used to conclude that the categories have the Krull-Schmidt property. This is only used to define the concept of rank of a map which is subsequently only used in Lemma 3.1.

In [17, Definition 9.3] it is shown that whenever the NilHecke algebra acts on 1-morphisms in an additive \mathbb{k} -linear idempotent complete 2-category \mathcal{K} one can define divided powers $E_i^{(a)} \mathbb{1}_\lambda$ as the image of a certain idempotents (with an appropriate shift). So the existence of these divided powers are a consequence of our definition rather than part of it.

1.3. Main results.

Definition 1.3. A 2-representation of $\dot{\mathcal{U}}_Q(\mathfrak{g})$ is a graded additive \mathbb{k} -linear 2-functor $\dot{\mathcal{U}}_Q(\mathfrak{g}) \rightarrow \mathcal{K}$ for some graded, additive 2-category \mathcal{K} .

When all of the Hom categories $\mathcal{K}(x, y)$ between objects x and y of \mathcal{K} are idempotent complete, in other words $Kar(\mathcal{K}) \cong \mathcal{K}$, then any graded additive \mathbb{k} -linear 2-functor $\mathcal{U}_Q(\mathfrak{g}) \rightarrow \mathcal{K}$ extends uniquely to a 2-representation of $\dot{\mathcal{U}}_Q(\mathfrak{g})$ (see section 2.1.2).

Theorem 1.1. Any Q -strong 2-representation of \mathfrak{g} on \mathcal{K} extends to a 2-representation $\mathcal{U}_Q(\mathfrak{g}) \rightarrow \mathcal{K}$.

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2. NOTATION AND DEFINITIONS

2.1. Some conventions. Fix a base field \mathbb{k} . We will always work over this field which is not assumed to be of characteristic 0, nor algebraically closed.

2.1.1. The Cartan datum and choice of scalars Q . We fix a Cartan datum consisting of

- a free \mathbb{Z} module X (the weight lattice),
- for $i \in I$ (I is an indexing set) there are elements $\alpha_i \in X$ (simple roots) and $\Lambda_i \in X$ (fundamental weights),
- for $i \in I$ an element $h_i \in X^\vee = \text{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (simple coroots),
- a bilinear form (\cdot, \cdot) on X .

Write $\langle \cdot, \cdot \rangle: X^\vee \times X \rightarrow \mathbb{Z}$ for the canonical pairing. These data should satisfy:

- $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ for any $i \in I$,
- $\langle i, \lambda \rangle := \langle h_i, \lambda \rangle = 2 \frac{(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $i \in I$ and $\lambda \in X$,
- $(\alpha_i, \alpha_j) \leq 0$ for $i, j \in I$ with $i \neq j$,
- $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $i, j \in I$.

Hence $C_{i,j} = \{\langle h_i, \alpha_j \rangle\}_{i,j \in I}$ is a symmetrizable generalized Cartan matrix. In what follows we write

$$(2.1) \quad d_{ij} = -\langle i, \alpha_j \rangle$$

for $i, j \in I$. We denote by $X^+ \subset X$ the dominant weights which are of the form $\sum_i \lambda_i \Lambda_i$ where $\lambda_i \geq 0$.

Write $d_i = \frac{(\alpha_i, \alpha_i)}{2}$ and let $q_i = q^{d_i}$, $[n]_i = q_i^{n-1} + q_i^{n-3} + \cdots + q_i^{1-n}$, $[n]_i! = [n]_i[n-1]_i \cdots [1]_i$.

Associated to a Cartan datum we also fix a choice of scalars Q consisting of

- t_{ij} for all $i, j \in I$,
- $s_{ij}^{pq} \in \mathbb{k}$ for $i \neq j$, $0 \leq p < d_{ij}$, and $0 \leq q < d_{ji}$,
- $r_i \in \mathbb{k}^\times$ for all $i \in I$,

such that

- $t_{ii} = 0$ for all $i \in I$ and $t_{ij} \in \mathbb{k}^\times$ for $i \neq j$,
- $s_{ij}^{pq} = s_{ji}^{qp}$,
- $t_{ij} = t_{ji}$ when $d_{ij} = 0$.

We set $s_{ij}^{pq} = 0$ when $p, q < 0$ or $d_{ij} \geq p$ or $d_{ji} \geq q$.

Given a set of scalars Q we denote by Q' another set of scalars given by

$$r'_i = -r_i, \quad t'_{ij} = t_{ji}^{-1} \quad i \neq j, \quad \text{and} \quad (s')_{ij}^{pq} = t_{ij}^{-1} t_{ji}^{-1} s_{ij}^{pq}.$$

2.1.2. 2-categories and 2-representations. By a graded category we will mean a category equipped with an auto-equivalence $\langle 1 \rangle$. We denote by $\langle l \rangle$ the auto-equivalence obtained by applying $\langle 1 \rangle$ l times. If A, B are two objects then $\text{Hom}^l(A, B)$ will be short hand for $\text{Hom}(A, B\langle l \rangle)$. A graded additive \mathbb{k} -linear 2-category is a category enriched over graded additive \mathbb{k} -linear categories, that is a 2-category \mathcal{K} such that the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ between objects A and B are graded additive \mathbb{k} -linear categories and the composition maps $\text{Hom}_{\mathcal{K}}(A, B) \times \text{Hom}_{\mathcal{K}}(B, C) \rightarrow \text{Hom}_{\mathcal{K}}(A, C)$ form a graded additive \mathbb{k} -linear functor.

A graded additive \mathbb{k} -linear 2-functor $F: \mathcal{K} \rightarrow \mathcal{K}'$ is a (weak) 2-functor that maps the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ to $\text{Hom}_{\mathcal{K}'}(FA, FB)$ by additive functors that commute with the auto-equivalence $\langle 1 \rangle$.

Given a 1-morphism A in an additive 2-category \mathcal{K} and a Laurent polynomial $f = \sum f_a q^a \in \mathbb{Z}[q, q^{-1}]$ we write $\oplus_f A$ or $A^{\oplus f}$ for the direct sum over $A \in \mathbb{Z}$, of f_a copies of $A\langle a \rangle$. In particular, if $f = [n]_i$, then we write $\oplus_{[n]_i} A$ to denote the direct sum $\oplus_{k=0}^{n-1} A\langle d_i(n-1-2k) \rangle$.

An additive category \mathcal{C} is said to be idempotent complete when every idempotent 1-morphism splits in \mathcal{C} . We say that the additive 2-category \mathcal{K} is idempotent complete when the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ are idempotent complete for any pair of objects A, B of \mathcal{K} , so that all idempotent 2-morphisms split. The idempotent completion, or Karoubi envelope $\text{Kar}(\mathcal{C})$, of an additive category \mathcal{C} can be viewed as a minimal enlargement of the category \mathcal{C} so that idempotents split. The idempotent completion $\text{Kar}(\mathcal{K})$ of a 2-category \mathcal{K} is the 2-category with the same objects as \mathcal{K} , but with Hom categories given by $\text{Kar}(\text{Hom}_{\mathcal{K}}(A, B))$. Any additive 2-functor $\mathcal{K} \rightarrow \mathcal{K}'$ that has splitting of idempotent 2-morphisms in \mathcal{K}' extends uniquely to an additive 2-functor $\text{Kar}(\mathcal{K}) \rightarrow \mathcal{K}'$, see [13, Section 3.4] for more details.

2.1.3. Conventions for \mathfrak{sl}_2 and Cartan datum with $I = \{i\}$. When $\mathfrak{g} = \mathfrak{sl}_2$ we use a simplified notation for the Cartan datum. We identify the weight lattice X with \mathbb{Z} by labeling the weight λ with $n \in \mathbb{Z}$ where $\langle i, \lambda \rangle = n$. In this case $d_{ii} = -\langle i, \alpha_i \rangle = 2$, $d_i = \frac{(\alpha_i, \alpha_i)}{2} = 1$, so that $q_i = q$ and $[n]_i = [n]$. Furthermore, the choice of scalars Q is determined by the single parameter r_i . By rescaling the ii -crossing, it is easy to see that for all choices of Q the 2-categories $\mathcal{U}_Q(\mathfrak{sl}_2)$ are isomorphic to the 2-category $\mathcal{U}(\mathfrak{sl}_2)$ given by taking $r_i = 1$.

For other Cartan datum with a single vertex $I = \{i\}$ and $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} + 2$, the 2-categories $\mathcal{U}_Q(\mathfrak{g})$ are again determined by the parameter r_i . By appropriate rescaling these 2-categories can be made to have $r_i = 1$. These 2-categories $\mathcal{U}_Q(\mathfrak{g})$ are isomorphic to the 2-category $\mathcal{U}(\mathfrak{g})$ by an additive \mathbb{k} -linear

functor that only effects the degrees of 2-morphisms. In this way, we can treat the case of a single vertex $I = \{i\}$ in a similar manner as the case of \mathfrak{sl}_2 .

2.2. Q -cyclic biadjointness. Diagrammatically, biadjointness corresponds to the following equalities of diagrams:

$$(2.2) \quad \begin{array}{c} \lambda + \alpha_i \\ \downarrow \\ \text{cap} \\ \downarrow \\ \lambda \end{array} = \begin{array}{c} \lambda + \alpha_i \\ \downarrow \\ \text{strand} \\ \downarrow \\ \lambda \end{array} \quad \begin{array}{c} \lambda + \alpha_i \\ \downarrow \\ \text{cap} \\ \downarrow \\ \lambda \end{array} = \begin{array}{c} \lambda + \alpha_i \\ \downarrow \\ \text{strand} \\ \downarrow \\ \lambda \end{array}$$

$$(2.3) \quad \begin{array}{c} \lambda \\ \downarrow \\ \text{cap} \\ \downarrow \\ \lambda + \alpha_i \end{array} = \begin{array}{c} \lambda \\ \downarrow \\ \text{strand} \\ \downarrow \\ \lambda + \alpha_i \end{array} \quad \begin{array}{c} \lambda \\ \downarrow \\ \text{cap} \\ \downarrow \\ \lambda + \alpha_i \end{array} = \begin{array}{c} \lambda \\ \downarrow \\ \text{strand} \\ \downarrow \\ \lambda + \alpha_i \end{array}$$

The Q -cyclic condition for dots is equivalent to:

$$(2.4) \quad \begin{array}{c} \lambda \\ \downarrow \\ \text{cap with dot} \\ \downarrow \\ \lambda + \alpha_i \end{array} = \begin{array}{c} \lambda \\ \downarrow \\ \text{strand with dot} \\ \downarrow \\ \lambda + \alpha_i \end{array} = \begin{array}{c} \lambda + \alpha_i \\ \downarrow \\ \text{cap with dot} \\ \downarrow \\ \lambda \end{array}$$

The Q -cyclic relations for crossings are given by

$$(2.5) \quad \begin{array}{c} \text{crossing} \\ \downarrow \\ \lambda \end{array} = t_{ij}^{-1} \begin{array}{c} \text{cap} \\ \downarrow \\ \text{strand} \\ \downarrow \\ \text{cap} \end{array} = t_{ji}^{-1} \begin{array}{c} \text{cap} \\ \downarrow \\ \text{strand} \\ \downarrow \\ \text{cap} \end{array}$$

This definition ensures that downward strands are equipped with an action of the KLR algebra associated with the choice of scalars Q' (we define Q' in section 2.1.1 and the KLR algebras in section 2.3).

The Q -cyclic condition for sideways crossings is given by the equalities:

$$(2.6) \quad \begin{array}{c} \text{sideways crossing} \\ \downarrow \\ \lambda \end{array} = \begin{array}{c} \text{cap} \\ \downarrow \\ \text{strand} \\ \downarrow \\ \text{cap} \end{array} = t_{ij} \begin{array}{c} \text{cap} \\ \downarrow \\ \text{strand} \\ \downarrow \\ \text{cap} \end{array}$$

$$(2.7) \quad \begin{array}{c} \text{sideways crossing} \\ \downarrow \\ \lambda \end{array} = \begin{array}{c} \text{cap} \\ \downarrow \\ \text{strand} \\ \downarrow \\ \text{cap} \end{array} = t_{ji} \begin{array}{c} \text{cap} \\ \downarrow \\ \text{strand} \\ \downarrow \\ \text{cap} \end{array}$$

where the second equality in (2.6) and (2.7) follow from (2.5).

2.3. KLR algebras. The KLR algebra $R = R_Q$ associated to Q is defined by finite \mathbb{k} -linear combinations of braid-like diagrams in the plane, where each strand is labeled by a vertex $i \in I$. Strands can intersect and can carry dots but triple intersections are not allowed. Diagrams are considered up to planar isotopy that do not change the combinatorial type of the diagram. We recall the local relations:

i) If all strands are labeled by the same $i \in I$ then the NilHecke algebra axioms hold

$$(2.8) \quad \begin{array}{c} \text{crossing} \end{array} = 0, \quad \begin{array}{c} \text{braid} \end{array} = \begin{array}{c} \text{braid} \end{array}$$

$$(2.9) \quad r_i \begin{array}{c} | \quad | \end{array} = \begin{array}{c} \text{crossing with dot on left} \end{array} - \begin{array}{c} \text{crossing with dot on right} \end{array} = \begin{array}{c} \text{crossing with dot on left} \end{array} - \begin{array}{c} \text{crossing with dot on right} \end{array}$$

ii) For $i \neq j$

$$(2.10) \quad \begin{array}{c} \text{crossing} \end{array} = \begin{cases} t_{ij} \begin{array}{c} | \quad | \end{array} & \text{if } (\alpha_i, \alpha_j) = 0, \\ t_{ij} \begin{array}{c} \text{dot on } i \end{array} \begin{array}{c} | \quad | \end{array} + t_{ji} \begin{array}{c} | \quad | \end{array} \begin{array}{c} \text{dot on } j \end{array} + \sum_{p,q} s_{ij}^{pq} \begin{array}{c} \text{dot on } i \end{array} \begin{array}{c} \text{dot on } j \end{array} & \text{if } (\alpha_i, \alpha_j) \neq 0, \end{cases}$$

where the summation in the case $(\alpha_i, \alpha_j) \neq 0$ is over all p, q such that

$$(2.11) \quad (\alpha_i, \alpha_i)p + (\alpha_j, \alpha_j)q = -2(\alpha_i, \alpha_j).$$

iii) For $i \neq j$ the dot sliding relations

$$(2.12) \quad \begin{array}{c} \text{crossing with dot on } i \end{array} = \begin{array}{c} \text{crossing with dot on } j \end{array} \quad \begin{array}{c} \text{crossing with dot on } j \end{array} = \begin{array}{c} \text{crossing with dot on } i \end{array}$$

hold.

iv) Unless $i = k$ and $(\alpha_i, \alpha_j) < 0$ the relation

$$(2.13) \quad \begin{array}{c} \text{braid} \end{array} = \begin{array}{c} \text{braid} \end{array}$$

holds. Otherwise, $(\alpha_i, \alpha_j) < 0$ and

$$(2.14) \quad r_i^{-1} \left(\begin{array}{c} \text{braid} \end{array} - \begin{array}{c} \text{braid} \end{array} \right) = t_{ij} \sum_{\ell_1 + \ell_2 = d_{ij} - 1} \begin{array}{c} \text{dot on } i \end{array} \begin{array}{c} | \quad | \end{array} \begin{array}{c} \text{dot on } j \end{array} + \sum_{p,q} s_{ij}^{pq} \sum_{\ell_1 + \ell_2 = p-1} \begin{array}{c} \text{dot on } i \end{array} \begin{array}{c} \text{dot on } j \end{array} \begin{array}{c} \text{dot on } i \end{array}$$

where the p, q summation is as in (2.11).

Remark.

- (1) It is always possible to rescale the ii -crossing by r_i so that $r_i = 1$ in the definition above. In the literature it is common to see both $r_i = 1$ or $r_i = -1$.
- (2) The 2-category $\mathcal{U}_Q(\mathfrak{g})$ is a cyclic 2-category (as defined in [17]) if $t_{ij} = t_{ji}$ for all i, j with $d_{ij} > 0$.

- (3) When the Cartan datum is symmetric, i.e. simply-laced, so that $(\alpha_i, \alpha_i) = 2$ and $(\alpha_i, \alpha_j) = -1$, then the KLR algebra takes on a simplified form. In particular, $s_{ij}^{pq} = 0$ for all i, j and p, q so that the summations in (2.12) vanish and $d_{ij} = 1$ in (2.14) so that the right-hand side has only one term. Furthermore, if the graph corresponding to the symmetric Cartan datum is a tree, then it is always possible to rescale the coefficients so that $t_{ij} = t_{ji} = 1$, see [14].

Inductively applying the NilHecke dot slide relation gives the equation

$$(2.15) \quad \begin{array}{c} \text{dot on top} \\ \text{cross} \\ \text{dot on bottom} \end{array} \begin{array}{c} m \\ n \end{array} - \begin{array}{c} \text{dot on top} \\ \text{cross} \\ \text{dot on top} \end{array} \begin{array}{c} m \\ n \end{array} = \begin{array}{c} \text{dot on top} \\ \text{cross} \\ \text{dot on top} \end{array} \begin{array}{c} m \\ n \end{array} - \begin{array}{c} \text{dot on bottom} \\ \text{cross} \\ \text{dot on bottom} \end{array} \begin{array}{c} m \\ n \end{array} = r_i \sum_{f_1+f_2=m-1} \begin{array}{c} \uparrow f_1 \\ \text{dot} \\ \uparrow f_2 \end{array} \begin{array}{c} n \end{array}$$

that will be useful in what follows.

2.4. Mixed relations. For $i \neq j$ relations

$$(2.16) \quad \begin{array}{c} \text{cross} \\ i \quad j \end{array}^\lambda = t_{ji} \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ j \end{array}^\lambda \quad \begin{array}{c} \text{cross} \\ i \quad j \end{array}^\lambda = t_{ij} \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ j \end{array}^\lambda$$

holds.

2.5. Dotted bubbles. Condition (4) of Definition 1.1 regarding dotted bubbles can be summarized diagrammatically as follows. Firstly, for all $m \in \mathbb{Z}_+$ one has

$$(2.17) \quad \begin{array}{c} \text{bubble} \\ i \quad \lambda \\ \text{dot} \\ m \end{array} = 0 \quad \text{if } m < \langle i, \lambda \rangle - 1, \quad \begin{array}{c} \text{bubble} \\ i \quad \lambda \\ \text{dot} \\ m \end{array} = 0 \quad \text{if } m < -\langle i, \lambda \rangle - 1$$

which means that dotted bubbles of negative degree are zero. On the other hand, a dotted bubble of degree zero is just the identity 2-morphism:

$$\begin{array}{c} \text{bubble} \\ i \quad \lambda \\ \text{dot} \\ \langle i, \lambda \rangle - 1 \end{array} = \text{Id}_{1_\lambda} \quad \text{for } \langle i, \lambda \rangle \geq 1, \quad \begin{array}{c} \text{bubble} \\ i \quad \lambda \\ \text{dot} \\ -\langle i, \lambda \rangle - 1 \end{array} = \text{Id}_{1_\lambda} \quad \text{for } \langle i, \lambda \rangle \leq -1.$$

2.6. Extended \mathfrak{sl}_2 relations. Here we use the conventions mentioned above for \mathfrak{sl}_2 (section 2.1.3).

In order to describe certain extended \mathfrak{sl}_2 relations it is convenient to use a shorthand notation from [17] called fake bubbles (see also section 5.1 for more details). These are diagrams for dotted bubbles where the labels of the number of dots is negative, but the total degree of the dotted bubble taken with these negative dots is still positive. They allow us to write these extended \mathfrak{sl}_2 relations more uniformly (i.e. independent on whether the weight n is positive or negative). The fake bubbles are defined inductively by the equations

$$(2.18) \quad \sum_{\ell_1+\ell_2=j} \begin{array}{c} \text{bubble} \\ n \\ \text{dot} \\ n-1+\ell_1 \end{array} \begin{array}{c} \text{bubble} \\ n \\ \text{dot} \\ -n-1+\ell_2 \end{array} = \delta_{j,0},$$

and the definition that in weight $n = 0$ we have

$$(2.19) \quad \begin{array}{c} \text{bubble} \\ 0 \\ \text{dot} \\ -1 \end{array} = \text{Id}_{1_0}, \quad \begin{array}{c} \text{bubble} \\ 0 \\ \text{dot} \\ -1 \end{array} = \text{Id}_{1_0}.$$

These equations arise from the homogeneous terms in t of the ‘infinite Grassmannian’ equation

$$(2.20) \quad \left(\begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1 \end{array} + \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1+1 \end{array} t + \cdots + \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1+\alpha \end{array} t^\alpha + \cdots \right) \left(\begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1 \end{array} + \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1+1 \end{array} t + \cdots + \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1+\alpha \end{array} t^\alpha + \cdots \right) = 1.$$

Now we can define the extended \mathfrak{sl}_2 relations.

If $n > 0$ then we have:

$$(2.21) \quad \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1 \end{array} = 0 \quad \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1 \end{array} = r_i \sum_{g_1+g_2=n} \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1+g_2 \end{array} \begin{array}{c} \text{vertical line with } g_1 \text{ dots} \end{array}$$

$$(2.22) \quad \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} = -r_i^{-2} \begin{array}{c} \text{crossing with } n \text{ dots} \end{array} + \sum_{\substack{f_1+f_2+f_3 \\ =n-1}} \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1+f_2 \end{array} \begin{array}{c} \text{vertical line with } f_1 \text{ dots} \end{array} \begin{array}{c} \text{vertical line with } f_3 \text{ dots} \end{array} \quad \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} = -r_i^{-2} \begin{array}{c} \text{crossing with } n \text{ dots} \end{array}$$

If $n < 0$ then we have:

$$(2.23) \quad \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1 \end{array} = -r_i \sum_{f_1+f_2=-n} \begin{array}{c} \text{vertical line with } f_1 \text{ dots} \end{array} \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1+f_2 \end{array} \quad \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1 \end{array} = 0$$

$$(2.24) \quad \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} = -r_i^{-2} \begin{array}{c} \text{crossing with } n \text{ dots} \end{array} \quad \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} = -r_i^{-2} \begin{array}{c} \text{crossing with } n \text{ dots} \end{array} + \sum_{\substack{g_1+g_2+g_3 \\ =-n-1}} \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1+g_2 \end{array} \begin{array}{c} \text{vertical line with } g_1 \text{ dots} \end{array} \begin{array}{c} \text{vertical line with } g_3 \text{ dots} \end{array}$$

If $n = 0$ then we have:

$$(2.25) \quad \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } -n-1 \end{array} = -r_i \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} \quad \begin{array}{c} \text{loop with } n \text{ dots} \\ \text{bottom dot at } n-1 \end{array} = r_i \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array}$$

$$(2.26) \quad \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} = -r_i^{-2} \begin{array}{c} \text{crossing with } n \text{ dots} \end{array} \quad \begin{array}{c} \text{vertical line with } n \text{ dots} \end{array} = -r_i^{-2} \begin{array}{c} \text{crossing with } n \text{ dots} \end{array}$$

Remark. Although these extended relations are a bit complicated, they are useful because they allow one to simplify any closed diagram so that it does not contain any loops or crossings.

2.7. An alternative characterization of the extended \mathfrak{sl}_2 relations. The extended \mathfrak{sl}_2 relations in section 2.6 are somewhat mysterious. On the other hand, there is an alternative, simpler way, to define or summarize them.

We will use the sideways crossings defined in (2.6) and (2.7). It turns out that the extended \mathfrak{sl}_2 conditions are equivalent to the following conditions:

- If $n \geq 0$ then the 2-morphism

$$(2.27) \quad \begin{array}{c} \text{sideways crossing} \end{array} \bigoplus_{k=0}^{n-1} \begin{array}{c} \text{loop with } k \text{ dots} \end{array} : \mathcal{FE}1_n \bigoplus_{k=0}^{n-1} 1_n \langle n-1-2k \rangle \rightarrow \mathcal{EF}1_n$$

is an isomorphism with the inverse given by

$$(2.28) \quad \left(\bigoplus_{k=0}^{n-1} \left(\sum_{\substack{j_1+j_2=j_1 \\ n-1-k}} \text{diagram} \right) : \mathcal{EF}\mathbf{1}_n \rightarrow \mathcal{FE}\mathbf{1}_n \bigoplus_{k=0}^{n-1} \mathbf{1}_n \langle n-1-2k \rangle \right)$$

- If $n \leq 0$ then the 2-morphism

$$(2.29) \quad \left(\bigoplus_{k=0}^{-n-1} \text{diagram} : \mathcal{EF}\mathbf{1}_n \bigoplus_{k=0}^{-n-1} \mathbf{1}_n \langle -n-1-2k \rangle \rightarrow \mathcal{FE}\mathbf{1}_n \right)$$

is an isomorphism with the inverse given by

$$(2.30) \quad \left(\bigoplus_{k=0}^{-n-1} \left(\sum_{\substack{j_1+j_2=j_1 \\ -n-1-k}} \text{diagram} \right) : \mathcal{FE}\mathbf{1}_n \rightarrow \mathcal{EF}\mathbf{1}_n \bigoplus_{k=0}^{-n-1} \mathbf{1}_n \langle -n-1-2k \rangle \right)$$

For more details see [19, Section 3].

3. SPACES OF MAPS AND BIADJOINTNESS

In this section we assume $\mathfrak{g} = \mathfrak{sl}_2$. By studying the spaces of maps between certain 1-morphisms we prove two main results. The first is that $\mathcal{E}s$ and $\mathcal{F}s$ are biadjoint. The second main result is Lemma 3.12 which describes $\text{Hom}^m(\mathcal{E}\mathbf{1}_n, \mathcal{E}\mathbf{1}_n)$ when $m \leq 2|n+1|$. This Lemma is an important tool used in the proof of Theorem 1.1.

3.1. Some general notions. The fact that the space of maps between any two 1-morphisms in a Q -strong 2-representation of \mathfrak{g} is finite dimensional means that the Krull-Schmidt property holds. This means that any 1-morphism has a unique direct sum decomposition (see section 2.2 of [21]). In particular, this means that if A, B, C are morphisms and V is a \mathbb{Z} -graded vector space then we have the following cancellation laws (see section 4 of [4]):

$$\begin{aligned} A \oplus B \cong A \oplus C &\Rightarrow B \cong C \\ A \otimes_{\mathbb{k}} V \cong B \otimes_{\mathbb{k}} V &\Rightarrow A \cong B. \end{aligned}$$

A brick in a (graded) category is an indecomposable object A such that $\text{End}(A) = \mathbb{k}$. For example, by Lemma 3.3 below, $\mathcal{E}\mathbf{1}_m$ is a brick.

Suppose that A is a brick and that X, Y are arbitrary objects. Then a morphism $f : X \rightarrow Y$ gives rise to a bilinear pairing $\text{Hom}(A, X) \times \text{Hom}(Y, A) \rightarrow \text{Hom}(A, A) = \mathbb{k}$. We define the A -rank $\text{rk}_A^0(f)$ of f to be the rank of this bilinear pairing.

We can also define A -rank as follows. Choose (non-canonical) direct sum decompositions $X = A \otimes V \oplus B$ and $Y = A \otimes V' \oplus B'$ where V, V' are \mathbb{k} vector spaces and B, B' do not contain A as a direct summand. Then one of the matrix coefficients of f is a map $A \otimes V \rightarrow A \otimes V'$, which (since A is a brick) is equivalent to a linear map $V \rightarrow V'$. The A -rank of f equals the rank of this linear map.

We define the total A -rank $\text{rk}_A(f)$ of f as $\sum_i \text{rk}_{A(i)}^0(f)$. In this paper, this will always turn out to be a finite integer. If the total A -rank of f is r then we say that “ f gives an isomorphism on r summands of the form $A\langle \cdot \rangle$ ”.

3.2. Induced maps.

Lemma 3.1. *If $n > 1$ then $\uparrow \downarrow : \mathbb{E}\mathbb{F}\mathbb{1}_n \rightarrow \mathbb{E}\mathbb{F}\mathbb{1}_n\langle 2 \rangle$ induces an isomorphism on $n - 1$ summands of the form $\mathbb{1}_n\langle k \rangle$, in other words $\text{rk}_{\mathbb{1}_n}(\uparrow \downarrow) = n - 1$.*

Similarly, if $n < -1$ then $\downarrow \uparrow : \mathbb{F}\mathbb{E}\mathbb{1}_n \rightarrow \mathbb{F}\mathbb{E}\mathbb{1}_n\langle 2 \rangle$ induces an isomorphism on $-n - 1$ summands of the form $\mathbb{1}_n\langle k \rangle$, in other words $\text{rk}_{\mathbb{1}_n}(\downarrow \uparrow) = -n - 1$.

Remark. If $n = -1, 0, 1$ the statement above is vacuous (which is why we only consider $n > 1$ and $n < -1$).

Proof. We prove the case $n > 1$ (the case $n < -1$ is proved in the same way). To do this we only use the commutator \mathfrak{sl}_2 relation together with the fact that $\uparrow \uparrow : \mathbb{E}\mathbb{E}\mathbb{1}_n \rightarrow \mathbb{E}\mathbb{E}\mathbb{1}_n\langle 2 \rangle$ induces a map

$$\mathbb{E}^{(2)}\mathbb{1}_n\langle -1 \rangle \oplus \mathbb{E}^{(2)}\mathbb{1}_n\langle 1 \rangle \rightarrow \mathbb{E}^{(2)}\mathbb{1}_n\langle 1 \rangle \oplus \mathbb{E}^{(2)}\mathbb{1}_n\langle 3 \rangle$$

which is an isomorphism on the common summand $\mathbb{E}^{(2)}\langle 1 \rangle$ (i.e. $\text{rk}_{\mathbb{E}^{(2)}}(\uparrow \uparrow) = 1$). This is a consequence of the NilHecke algebra action.

Using this fact, the map

$$(3.1) \quad \uparrow \uparrow \downarrow : \mathbb{E}\mathbb{E}\mathbb{F}\mathbb{1}_{n-2} \rightarrow \mathbb{E}\mathbb{E}\mathbb{F}\mathbb{1}_{n-2}\langle 2 \rangle.$$

induces an isomorphism on the summand $\mathbb{E}^{(2)}\mathbb{F}\mathbb{1}_{n-2}\langle 1 \rangle$. Now, by [17, Proposition 9.6] or [5, Lemma 4.2] we know that

$$\mathbb{E}^{(2)}\mathbb{F}\mathbb{1}_{n-2} \cong \mathbb{F}^{(2)}\mathbb{E}\mathbb{1}_{n-2} \bigoplus_{i=0}^{n-2} \mathbb{E}\mathbb{1}_{n-2}\langle n-2-2i \rangle$$

which means that $\text{rk}_{\mathbb{E}}(\uparrow \uparrow \downarrow) \geq n - 1$.

On the other hand, by decomposing $\mathbb{E}\mathbb{F}\mathbb{1}_{n-2}$, the map in (3.1) induces the map

$$\uparrow \downarrow \uparrow \bigoplus_{i=0}^{n-3} \uparrow : \mathbb{E}\mathbb{F}\mathbb{E}\mathbb{1}_{n-2} \bigoplus_{i=0}^{n-3} \mathbb{E}\mathbb{1}_{n-2}\langle n-3-2i \rangle \rightarrow \mathbb{E}\mathbb{F}\mathbb{E}\mathbb{1}_{n-2}\langle 2 \rangle \bigoplus_{i=0}^{n-3} \mathbb{E}\mathbb{1}_{n-2}\langle n-1-2i \rangle.$$

Thus the total \mathbb{E} -rank of (3.1) is equal to the total \mathbb{E} -rank of

$$\uparrow \downarrow \uparrow : \mathbb{E}\mathbb{F}\mathbb{E}\mathbb{1}_{n-2} \rightarrow \mathbb{E}\mathbb{F}\mathbb{E}\mathbb{1}_{n-2}\langle 2 \rangle.$$

Since $\mathbb{E}\mathbb{F}\mathbb{E}\mathbb{1}_{n-2} \cong \mathbb{F}\mathbb{E}\mathbb{E}\mathbb{1}_{n-2} \bigoplus_{i=0}^{n-1} \mathbb{E}\mathbb{1}_{n-2}\langle n-1-2i \rangle$ where $\mathbb{F}\mathbb{E}\mathbb{E} = \mathbb{F}\mathbb{E}^{(2)}\langle -1 \rangle \oplus \mathbb{F}\mathbb{E}^{(2)}\langle 1 \rangle$ contains no summands of \mathbb{E} it follows that the total \mathbb{E} -rank of $\uparrow \downarrow : \mathbb{E}\mathbb{F}\mathbb{1}_n \rightarrow \mathbb{E}\mathbb{F}\mathbb{1}_n\langle 2 \rangle$ is at least $n - 1$. The result follows since, by degree reasons, it cannot be any bigger than this. \square

3.3. Biadjointness. Recall that in a Q -strong 2-representation we define $\mathbb{1}_n\mathbb{F}$ as the right adjoint of $\mathbb{E}\mathbb{1}_n$ together with a grading shift (1.1). We now show that \mathbb{E} s and \mathbb{F} s are biadjoint to each other (up to a grading shift).

Proposition 3.2. *Given a strong 2-representation of \mathfrak{sl}_2 the left and right adjoints of \mathbb{E} are isomorphic up to specified shifts. More precisely $(\mathbb{E}\mathbb{1}_n)_L \cong (\mathbb{E}\mathbb{1}_n)_R\langle -2(n+1) \rangle$ given by the 2-morphisms below which are uniquely determined up to a scalar and induce the adjunction maps.*

$$(3.2) \quad \curvearrowright : \mathbb{E}\mathbb{F}\mathbb{1}_{n+2}\langle n+1 \rangle \rightarrow \mathbb{1}_{n+2} \quad \curvearrowleft : \mathbb{1}_{n+2}\langle n+1 \rangle \rightarrow \mathbb{E}\mathbb{F}\mathbb{1}_{n+2}$$

$$(3.3) \quad \curvearrowleft : \mathbb{F}\mathbb{E}\mathbb{1}_n \rightarrow \mathbb{1}_n\langle n+1 \rangle \quad \curvearrowright : \mathbb{1}_n \rightarrow \mathbb{F}\mathbb{E}\mathbb{1}_n\langle n+1 \rangle.$$

Remark. We will choose specific scalars for the 2-morphisms above in section 4.1. The result above implies that

- (1) $(\mathbb{E}^{(r)}\mathbb{1}_n)_R \cong \mathbb{1}_n\mathbb{F}^{(r)}\langle r(n+r) \rangle$,
- (2) $(\mathbb{E}^{(r)}\mathbb{1}_n)_L \cong \mathbb{1}_n\mathbb{F}^{(r)}\langle -r(n+r) \rangle$.

We prove Proposition 3.2 when $n \geq 0$ (the case $n \leq 0$ is similar - see Remark 3.3.1). First, to define the 2-morphism \curvearrowright in (3.2) we note that

$$\begin{aligned}
 & \text{Hom}(\mathbb{1}_{n+2}\langle n+1 \rangle, \mathbf{E}\mathbb{F}\mathbb{1}_{n+2}) \\
 \cong & \text{Hom}(\mathbb{1}_{n+2}\langle n+1 \rangle, \mathbf{F}\mathbf{E}\mathbb{1}_{n+2} \bigoplus_{[n+2]} \mathbb{1}_{n+2}) \\
 \cong & \text{Hom}(\mathbb{1}_{n+2}\langle n+1 \rangle, (\mathbf{E}\mathbb{1}_{n+2})_R \mathbf{E}\mathbb{1}_{n+2}\langle -n-3 \rangle) \bigoplus_{k=0}^{n+2} \text{Hom}(\mathbb{1}_{n+2}, \mathbb{1}_{n+2}\langle -2k \rangle) \\
 \cong & \text{Hom}(\mathbf{E}\mathbb{1}_{n+2}, \mathbf{E}\mathbb{1}_{n+2}\langle -2n-4 \rangle) \oplus \text{Hom}(\mathbb{1}_n, \mathbb{1}_n)
 \end{aligned}$$

We take \curvearrowright to be the identity map in the second summand. This corresponds to the inclusion of $\mathbb{1}_{n+2}$ into the lowest degree summand of $\mathbb{1}_{n+2}$ inside $\mathbf{E}\mathbb{F}\mathbb{1}_{n+2}$. Similarly, we define the left hand map in (3.2) as the projection out of the top degree summand $\mathbb{1}_{n+2}$ in $\mathbf{E}\mathbb{F}\mathbb{1}_{n+2}$.

The right hand map in (3.3) is defined by adjunction using the fact that \mathbf{F} equals \mathbf{E}_R up to a shift. The last adjunction is more difficult to define and we will do this in the process of proving proposition 3.2. The proof is by decreasing induction on n starting from the highest weight. In particular, we use here that the representation is integrable.

Fix $n \geq 0$. By induction we can assume that if $m > n$ we have

$$(3.4) \quad (\mathbf{E}\mathbb{1}_m)_R \cong \mathbb{1}_m \mathbf{F}\langle m+1 \rangle \text{ and } (\mathbf{E}\mathbb{1}_m)_L \cong \mathbb{1}_m \mathbf{F}\langle -m-1 \rangle.$$

since when m is beyond the highest weight this claim is vacuously true since both maps are zero. Assuming the induction hypothesis (3.4) above we have the following facts. The reader should recall the ‘Important Convention’ from Definition 1.2.

Lemma 3.3. *If $m \geq n$ then $\text{Hom}(\mathbf{E}\mathbb{1}_m, \mathbf{E}\mathbb{1}_m\langle l \rangle)$ is zero if $l < 0$ and one-dimensional if $l = 0$ and likewise for $\text{Hom}(\mathbb{1}_m \mathbf{F}, \mathbb{1}_m \mathbf{F}\langle l \rangle)$.*

Proof. We prove the result for \mathbf{E} by (decreasing) induction on m (the result for \mathbf{F} follows by adjunction). We have

$$\begin{aligned}
 & \text{Hom}(\mathbf{E}\mathbb{1}_m, \mathbf{E}\mathbb{1}_m\langle l \rangle) \\
 \cong & \text{Hom}(\mathbf{E}(\mathbf{E}\mathbb{1}_m)_R \mathbb{1}_{m+2}, \mathbb{1}_{m+2}\langle l \rangle) \\
 \cong & \text{Hom}(\mathbf{E}\mathbb{F}\mathbb{1}_{m+2}\langle m+1 \rangle, \mathbb{1}_{m+2}\langle l \rangle) \\
 \cong & \text{Hom}(\mathbf{F}\mathbf{E}\mathbb{1}_{m+2} \bigoplus_{[m+2]} \mathbb{1}_{m+2}, \mathbb{1}_{m+2}\langle l-m-1 \rangle) \\
 \cong & \text{Hom}((\mathbf{E}\mathbb{1}_{m+2})_L \mathbf{E}\mathbb{1}_{m+2}\langle m+3 \rangle, \mathbb{1}_{m+2}\langle l-m-1 \rangle) \oplus \text{Hom}(\mathbb{1}_{m+2}, \bigoplus_{[m+2]} \mathbb{1}_{m-2}\langle l-m-1 \rangle) \\
 \cong & \text{Hom}(\mathbf{E}\mathbb{1}_{m+2}, \mathbf{E}\mathbb{1}_{m+2}\langle l-2m-4 \rangle) \bigoplus_{k=0}^{m+1} \text{Hom}(\mathbb{1}_{m+2}, \mathbb{1}_{m+2}\langle l-2k \rangle).
 \end{aligned}$$

By induction the first term above is zero and, by condition (2) of Definition 1.2, all the terms in the direct sum are zero unless $k = 0$ and $l = 0$. In that case we get $\text{Hom}(\mathbb{1}_{m+2}, \mathbb{1}_{m+2}) \cong \mathbb{k}$ and we are done. \square

Corollary 3.4. *If $m \geq n-2$ then $\text{Hom}(\mathbf{E}\mathbf{E}\mathbb{1}_m, \mathbf{E}\mathbf{E}\mathbb{1}_m\langle l \rangle)$ is zero if $l < -2$ and one-dimensional if $l = -2$.*

Proof. The proof is by (decreasing) induction on m . We have

$$\begin{aligned}
& \text{Hom}(\mathbb{E}\mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{E}\mathbb{1}_m) \\
& \cong \text{Hom}(\mathbb{E}\mathbb{E}(\mathbb{E}\mathbb{1}_m)_R \mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}) \\
& \cong \text{Hom}(\mathbb{E}\mathbb{E}\mathbb{F}\mathbb{1}_{m+2}\langle m+1 \rangle, \mathbb{E}\mathbb{1}_{m+2}) \\
& \cong \text{Hom}(\mathbb{E}\mathbb{F}\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}\langle -m-1 \rangle) \bigoplus_{k=0}^{m+1} \text{Hom}(\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}\langle m+1-2k-m-1 \rangle) \\
& \cong \text{Hom}(\mathbb{F}\mathbb{E}\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}\langle -m-1 \rangle) \bigoplus_{k=0}^{m+3} \text{Hom}(\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}\langle m+3-2k-m-1 \rangle) \\
& \quad \bigoplus_{k=0}^{m+1} \text{Hom}(\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}\langle -2k \rangle) \\
& \cong \text{Hom}(\mathbb{E}\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{E}\mathbb{1}_{m+2}\langle -2m-6 \rangle) \bigoplus_{k=0}^{m+3} \text{Hom}(\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}_{m+2}\langle -2k+2 \rangle) \\
& \quad \bigoplus_{k=0}^{m+1} \text{Hom}(\mathbb{E}\mathbb{1}_{m+2}, \mathbb{E}\mathbb{1}_{m+2}\langle -2k \rangle).
\end{aligned}$$

Shifting by $\langle l \rangle$ where $l < -2$ we find that the first term is zero by induction and the others are zero by Lemma 3.3. If $l = -2$ the same vanishing holds with the exception of the term in the middle summation when $k = 0$ which yields $\text{End}(\mathbb{E}\mathbb{1}_{m+2}) \cong \mathbb{k}$. \square

Corollary 3.5. *If $m \geq n$ then $\text{Hom}(\mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{E}\mathbb{F}\mathbb{1}_m\langle -m-1 \rangle) \cong \mathbb{k}$.*

Proof. Moving the \mathbb{F} past the \mathbb{E} 's using the \mathfrak{sl}_2 commutation relation we get

$$\begin{aligned}
& \text{Hom}(\mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{E}\mathbb{F}\mathbb{1}_m\langle -m-1 \rangle) \\
& \cong \text{Hom}(\mathbb{E}\mathbb{1}_m, \mathbb{F}\mathbb{E}\mathbb{E}\mathbb{1}_m\langle -m-1 \rangle) \bigoplus_{k=0}^{m+1} \text{Hom}(\mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{1}_m\langle -2k \rangle) \bigoplus_{k=0}^{m-1} \text{Hom}(\mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{1}_m\langle -2k-2 \rangle).
\end{aligned}$$

By Lemma 3.3 all the terms in middle and right summations are zero except in the middle summation when $k = 0$ in which case we get $\text{End}(\mathbb{E}\mathbb{1}_m) \cong \mathbb{k}$. The term on the left is equal to

$$\text{Hom}((\mathbb{1}_{m+2}\mathbb{F})_L \mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{E}\mathbb{1}_m\langle -m-1 \rangle) \cong \text{Hom}(\mathbb{E}\mathbb{E}\mathbb{1}_m, \mathbb{E}\mathbb{E}\mathbb{1}_m\langle -2m-4 \rangle)$$

which vanishes by Corollary 3.4 since $n \geq 0$. \square

Corollary 3.6. *The maps*

$$(3.5) \quad \begin{array}{c} n \\ \circlearrowleft \\ n-1 \end{array} : \mathbb{1}_n \rightarrow \mathbb{1}_n \quad \text{for } n > 0,$$

$$(3.6) \quad \begin{array}{c} n \\ \circlearrowright \\ -n-1 \end{array} : \mathbb{1}_n \rightarrow \mathbb{1}_n \quad \text{for } n < 0,$$

are both equal to (possibly different) non-zero scalars of $\text{Id}_{\mathbb{1}_n}$.

Proof. As usual, we prove the case $n > 0$ (the case $n < 0$ follows similarly).

By construction the map $\smile: \mathbb{1}_n \rightarrow \text{EF}\mathbb{1}_n\langle -n+1 \rangle$ is an isomorphism between the summand $\mathbb{1}_n$ on the left and the corresponding $\mathbb{1}_n$ on the right hand side. Then applying ${}^{n-1}\downarrow: \text{EF}\mathbb{1}_n\langle -n+1 \rangle \rightarrow \text{EF}\mathbb{1}_n\langle n-1 \rangle$ induces an isomorphism between the summands $\mathbb{1}_n$ on either side (this is a corollary of Lemma 3.1).

Finally, again by construction, the map $\cap: \text{EF}\mathbb{1}_n\langle n-1 \rangle \rightarrow \mathbb{1}_n$ is an isomorphism between $\mathbb{1}_n$ on the right side and the corresponding summand $\mathbb{1}_n$ on the left hand side. Thus the composition

$$\mathbb{1}_n \xrightarrow{\smile} \text{EF}\mathbb{1}_n\langle -n+1 \rangle \xrightarrow{{}^{n-1}\downarrow} \text{EF}\mathbb{1}_n\langle n-1 \rangle \xrightarrow{\cap} \mathbb{1}_n$$

is an isomorphism and this completes the proof since $\text{Hom}(\mathbb{1}_n, \mathbb{1}_n) \cong \mathbb{k}$. \square

3.3.1. Defining the last adjunction. We are still assuming, for simplicity, that $n \geq 0$. Recall that we defined all but one of the adjunction maps, namely the 2-morphism \cap in (3.3). This map cannot be defined formally by adjunction and since $n \geq 0$ the 1-morphism $\text{FE}\mathbb{1}_n$ is indecomposable so one cannot define it as an inclusion. To overcome this problem we construct this 2-morphism by defining an up-down crossing and composing it with the left map in (3.2).

We define the crossing \times as the map $\text{FE}\mathbb{1}_n \rightarrow \text{EF}\mathbb{1}_n \cong \text{FE}\mathbb{1}_n \oplus_{[n]} \mathbb{1}_n$ which includes $\text{FE}\mathbb{1}_n$ into $\text{EF}\mathbb{1}_n$. Note that this map is not unique because there exist non-zero maps $\text{FE}\mathbb{1}_n \rightarrow \oplus_{[n]} \mathbb{1}_n$ but this ambiguity will not matter. We then define the 2-morphism \cap as the composite

$$(3.7) \quad \text{FE}\mathbb{1}_n \xrightarrow{\times} \text{EF}\mathbb{1}_n \xrightarrow{{}^n\downarrow} \text{EF}\mathbb{1}_n\langle 2n \rangle \xrightarrow{\cap} \mathbb{1}_n\langle n+1 \rangle.$$

A similar definition of the cap 2-morphism appears in [17, Proposition 5.4] and [22, Section 4.1.4].

Lemma 3.7. *If $m \geq n$ then the following two maps*

$$\uparrow \times \quad m \qquad \times \downarrow \quad m$$

are non-zero multiples of the unique 2-morphism $\text{E}\mathbb{1}_m \rightarrow \text{EEF}\mathbb{1}_m\langle -m-1 \rangle$ from Corollary 3.5.

Proof. We just need to show that both maps are non-zero. Suppose the map on the right is zero. Adding a dot at the top of the middle upward pointing strand and sliding it past the crossing using the NilHecke relation (2.9) one gets two terms. One term is again zero (because it is the composition of a dot and the original map) and the other is just the adjoint map

$$\text{E}\mathbb{1}_m \xrightarrow{\uparrow \smile} \text{EEF}\mathbb{1}_m\langle -m+1 \rangle$$

which cannot be zero because it is the inclusion of $\text{E}\mathbb{1}_m$ into the lowest degree summand inside $\text{EEF}\mathbb{1}_m\langle -m+1 \rangle$. Thus the map on the right must be non-zero.

On the other hand, the map on the left is the composition

$$\text{E}\mathbb{1}_m \xrightarrow{\smile \uparrow} \text{EFE}\mathbb{1}_m\langle -m-1 \rangle \xrightarrow{\uparrow \times} \text{EEF}\mathbb{1}_m\langle -m-1 \rangle$$

where the first map is an inclusion of $\text{E}\mathbb{1}_m$ into the lowest degree copy of $\text{E}\mathbb{1}_m$ in $(\text{EF})\text{E}\mathbb{1}_m$ and the second map is induced by the inclusion of $\text{EF}\mathbb{1}_{m+2}$ into $\text{FE}\mathbb{1}_{m+2} \cong \text{EF}\mathbb{1}_{m+2} \oplus_{[m+2]} \mathbb{1}_{m+2}$. Since this is the composition of two inclusions it is also non-zero. \square

Proposition 3.8. *The 2-morphisms*

$$\begin{aligned} \cap : \mathbf{EF}\mathbb{1}_{n+2}\langle n+1 \rangle &\rightarrow \mathbb{1}_{n+2} & \text{and} & & \cup : \mathbb{1}_{n+2}\langle n+1 \rangle &\rightarrow \mathbf{FE}\mathbb{1}_{n+2} \\ \cup : \mathbb{1}_{n+2}\langle n+1 \rangle &\rightarrow \mathbf{EF}\mathbb{1}_{n+2} & \text{and} & & \cap : \mathbf{FE}\mathbb{1}_n &\rightarrow \mathbb{1}_n\langle n+1 \rangle \end{aligned}$$

defined above satisfy the adjunction relations (2.2) and (2.3) up to non-zero multiples.

Proof. We prove the adjunction map on the left of (2.2) (the second one follows formally). Since $\text{End}(\mathbb{1}_n) \cong \mathbb{k}$ it suffices to show that the left side of (2.2) is non-zero. Now, $\text{Hom}(\mathbf{EF}\mathbb{1}_{n+2}\langle n+1 \rangle, \mathbb{1}_{n+2}) \cong \text{Hom}(\mathbf{E}\mathbb{1}_n, \mathbf{E}\mathbb{1}_n) \cong \mathbb{k}$ by Lemma 3.3. So the map $\cap : \mathbf{EF}\mathbb{1}_{n+2}\langle n+1 \rangle \rightarrow \mathbb{1}_{n+2}$ must be equal to the adjunction map (up to a multiple). Since $\cup : \mathbb{1}_{n+2}\langle n+1 \rangle \rightarrow \mathbf{FE}\mathbb{1}_{n+2}$ is defined to be the adjunction map their composition must be non-zero.

Likewise, we now prove the adjunction map on the left in (2.3) (the second one follows formally) by showing that the left side of (2.3) is non-zero. Using Lemma 3.7 it follows that up to non-zero multiples the left side of (2.3) is equal to the composition

$$(3.8) \quad \text{Diagram: A crossing of two strands with a dot on the upper-left strand. The top-left strand is labeled 'n' and the top-right strand is labeled 'n'. The diagram represents a composition of morphisms.$$

Moving one of the dots through the crossing using the NilHecke relation (2.9) gives

$$(3.9) \quad \text{Diagram: Crossing with dot on upper-left strand, labeled 'n' and 'n' on top strands.} = \text{Diagram: Crossing with dot on lower-left strand, labeled 'n-1' and 'n' on top strands.} + \text{Diagram: A dot on a strand labeled 'n-1' with a loop labeled 'n' on top and 'n-1' on bottom.$$

where the first term on the right hand side is zero since it is the composite of a dot and an endomorphism of $\mathbf{E}\mathbb{1}_n$ of degree -2 . By Corollary 3.6 the remaining term on the right-hand side above is some non-zero multiple of the identity. \square

Thus we get that

$$(\mathbf{E}\mathbb{1}_n)_R \cong \mathbb{1}_n \mathbf{F}\langle n+1 \rangle \text{ and } (\mathbf{E}\mathbb{1}_n)_L \cong \mathbb{1}_n \mathbf{F}\langle -n-1 \rangle.$$

which completes the induction. To finish off the proof of Proposition 3.2 we show that these adjunction maps are unique up to a multiple.

Corollary 3.9. *Given a strong 2-representation of \mathfrak{sl}_2 we have*

$$\begin{aligned} \text{Hom}(\mathbf{EF}\mathbb{1}_{n+2}\langle n+1 \rangle, \mathbb{1}_{n+2}) &\cong \mathbb{k}, & \text{Hom}(\mathbb{1}_{n+2}\langle n+1 \rangle, \mathbf{EF}\mathbb{1}_{n+2}) &\cong \mathbb{k}, \\ \text{Hom}(\mathbf{FE}\mathbb{1}_n, \mathbb{1}_n\langle n+1 \rangle) &\cong \mathbb{k}, & \text{Hom}(\mathbb{1}_n, \mathbf{FE}\mathbb{1}_n\langle n+1 \rangle) &\cong \mathbb{k}. \end{aligned}$$

Proof. We calculate the first space (the other three are similar). We have

$$\begin{aligned} \text{Hom}(\mathbf{EF}\mathbb{1}_{n+2}\langle n+1 \rangle, \mathbb{1}_{n+2}) &\cong \text{Hom}(\mathbf{F}\mathbb{1}_{n+2}\langle n+1 \rangle, (\mathbf{E}\mathbb{1}_n)_R \mathbb{1}_{n+2}) \\ &\cong \text{Hom}(\mathbf{F}\mathbb{1}_{n+2}, \mathbf{F}\mathbb{1}_{n+2}) \\ &\cong \mathbb{k}. \end{aligned}$$

\square

Remark. The proof of Proposition 3.2 when $n \leq 0$ is analogous to the one above. But since there is a break in symmetry in the definition of a Q -strong 2-representation of \mathfrak{g} , namely we use \mathbf{E}_R instead of \mathbf{E}_L , we spell out a few details about how to alter the proof above.

We fix $n \leq 0$. The induction hypothesis (3.4) is now for $m < n$. Lemma 3.3 is now for $m \leq n$ and one proves it by moving the E by adjunction from the left side to the right side. Likewise, Corollary 3.4 is for $m \leq n + 2$.

Now, the left map in (3.2) is defined by adjunction. The maps in (3.3) are defined using the decomposition of $FE\mathbb{1}_n$ (before this defined both maps in (3.2)). Finally, the right map in (3.2) is defined by the composition

$$\mathbb{1}_{n+2} \xrightarrow{\quad \cup \quad} FE\mathbb{1}_{n+2}\langle n+3 \rangle \xrightarrow{\quad \downarrow \uparrow -n-2 \quad} FE\mathbb{1}_{n+2}\langle -n-1 \rangle \xrightarrow{\quad \times \quad} EF\mathbb{1}_{n+2}\langle -n-1 \rangle.$$

3.4. Up-down crossings. At this point we know that both the left and right adjoint of F is E (up to a specified shift). Consequently we can prove the following.

Lemma 3.10. *For any $n \in \mathbb{Z}$ we have*

$$\mathrm{Hom}(EF\mathbb{1}_n, FE\mathbb{1}_n) \cong \mathbb{k} \cong \mathrm{Hom}(FE\mathbb{1}_n, EF\mathbb{1}_n).$$

Proof. We prove that $\mathrm{Hom}(EF\mathbb{1}_n, FE\mathbb{1}_n) \cong \mathbb{k}$ (the other case follows similarly). One has

$$\begin{aligned} \mathrm{Hom}(EF\mathbb{1}_n, FE\mathbb{1}_n) &\cong \mathrm{Hom}(F(E\mathbb{1}_n)_R \mathbb{1}_{n+2}, (E\mathbb{1}_{n-2})_R F\mathbb{1}_{n+2}) \\ &\cong \mathrm{Hom}(FF\mathbb{1}_{n+2}\langle n+1 \rangle, FF\mathbb{1}_{n+2}\langle n-1 \rangle) \cong \mathbb{k} \end{aligned}$$

where the last isomorphism follows from (the adjoint of) Corollary 3.4. \square

Subsequently, we denote these maps $FE\mathbb{1}_n \rightarrow EF\mathbb{1}_n$ and $EF\mathbb{1}_n \rightarrow FE\mathbb{1}_n$ by \times and \times respectively. For the moment these maps are uniquely defined only up to a non-zero scalar.

Corollary 3.11. *If $n \geq 0$ then the map*

$$(3.10) \quad \zeta := \times \bigoplus_{k=0}^{n-1} \mathbb{1}_k \cup : FE\mathbb{1}_n \bigoplus_{k=0}^{n-1} \mathbb{1}_n\langle n-1-2k \rangle \rightarrow EF\mathbb{1}_n$$

induces an isomorphism. Likewise, if $n \leq 0$ then the map

$$(3.11) \quad \zeta := \times \bigoplus_{k=0}^{-n-1} \mathbb{1}_k \cup : EF\mathbb{1}_n \bigoplus_{k=0}^{-n-1} \mathbb{1}_n\langle -n-1-2k \rangle \rightarrow FE\mathbb{1}_n$$

is an isomorphism.

Proof. We prove the case $n \geq 0$ (the case $n \leq 0$ is proved similarly).

We know that

$$EF\mathbb{1}_n \cong FE\mathbb{1}_n \bigoplus_{k=0}^{n-1} \mathbb{1}_n\langle n-1-2k \rangle$$

and by Lemma 3.10 the map \times must induce an isomorphism between the $FE\mathbb{1}_n$ summands and must induce the zero map from the $FE\mathbb{1}_n$ summand on the left to any summand $\mathbb{1}_n\langle -n-1+2k \rangle$ on the right.

It remains to show that $\bigoplus_{k=0}^{n-1} \mathbb{1}_k \cup$ induces an isomorphism between the summands $\mathbb{1}_n\langle n-1-2k \rangle$ on either side. Since $\mathrm{Hom}(\mathbb{1}_n, \mathbb{1}_n\langle l \rangle) = 0$ if $l < 0$ it follows that the induced map

$$\bigoplus_{k=0}^{n-1} \mathbb{1}_n\langle n-1-2k \rangle \rightarrow \bigoplus_{k=0}^{n-1} \mathbb{1}_n\langle n-1-2k \rangle$$

is upper triangular (when expressed as a matrix). It remains to show that the maps on the diagonal are isomorphisms between the summands $\mathbb{1}_n\langle n-1-2k \rangle$ on either side.

Now, by construction, the map $\smile : \mathbb{1}_n \langle n-1 \rangle \rightarrow \mathbf{E}\mathbb{F}\mathbb{1}_n$ is an isomorphism onto the summand $\mathbb{1}_n \langle n-1 \rangle$ on the right side. Consequently, by Lemma 3.1, the composition

$$\mathbb{1}_n \langle n-1-2k \rangle \xrightarrow{\smile} \mathbf{E}\mathbb{F}\mathbb{1}_n \langle -2k \rangle \xrightarrow{\begin{smallmatrix} \uparrow k \\ \downarrow \end{smallmatrix}} \mathbf{E}\mathbb{F}\mathbb{1}_n$$

must also induce an isomorphism between the summands $\mathbb{1}_n \langle n-1-2k \rangle$ on either side. This proves that all the diagonal entries are isomorphisms so we are done. \square

3.5. Endomorphisms of $\mathbf{E}\mathbb{1}_n$.

Lemma 3.12. *Suppose $m < 2|n+2|$ (or $m = 2$ and $n = -1$) and $f \in \text{Hom}^m(\mathbf{E}\mathbb{1}_n, \mathbf{E}\mathbb{1}_n)$. If $n \geq -1$ then f is of the form*

$$(3.12) \quad \sum_i \begin{array}{c} n+2 \\ \boxed{f_i} \end{array} \begin{array}{c} \uparrow \\ \bullet \\ i \end{array} \begin{array}{c} n \end{array}$$

where $f_i \in \text{Hom}^{m-2i}(\mathbb{1}_{n+2}, \mathbb{1}_{n+2})$. Similarly, if $n \leq -1$ (or $m = 2$ and $n = -1$) then f is of the form

$$(3.13) \quad \sum_i \begin{array}{c} n+2 \\ \uparrow \\ \bullet \\ i \end{array} \begin{array}{c} n \\ \boxed{f_i} \end{array}$$

where $f_i \in \text{Hom}^{m-2i}(\mathbb{1}_n, \mathbb{1}_n)$. By adjunction there are analogous results for $f \in \text{Hom}^m(\mathbf{F}\mathbb{1}_n, \mathbf{F}\mathbb{1}_n)$.

Proof. We prove the case $n \geq -1$ (the case $n \leq -1$ is proved similarly). We will deal with the special case $m = 2$ and $n = -1$ at the end. We have

$$\begin{aligned} \text{Hom}^m(\mathbf{E}\mathbb{1}_n, \mathbf{E}\mathbb{1}_n) &\cong \text{Hom}^m(\mathbb{1}_{n+2}, \mathbf{E}(\mathbf{E}\mathbb{1}_n)_L \mathbb{1}_{n+2}) \\ &\cong \text{Hom}^m(\mathbb{1}_{n+2}, \mathbf{E}\mathbb{F}\mathbb{1}_{n+2} \langle -(n+1) \rangle) \\ &\cong \text{Hom}^m(\mathbb{1}_{n+2}, \bigoplus_{[n+2]} \mathbb{1}_{n+2} \langle -(n+1) \rangle \oplus \mathbf{F}\mathbf{E}\mathbb{1}_{n+2} \langle -(n+1) \rangle) \\ &\cong \text{Hom}^{m-n-1}(\mathbb{1}_{n+2}, \bigoplus_{[n+2]} \mathbb{1}_{n+2}) \oplus \text{Hom}^m((\mathbf{F}\mathbb{1}_{n+4})_L, \mathbf{E}\mathbb{1}_{n+2} \langle -(n+1) \rangle) \\ &\cong \text{Hom}^{m-n-1}(\mathbb{1}_{n+2}, \bigoplus_{[n+2]} \mathbb{1}_{n+2}) \oplus \text{Hom}^m(\mathbf{E}\mathbb{1}_{n+2} \langle n+3 \rangle, \mathbf{E}\mathbb{1}_{n+2} \langle -(n+1) \rangle) \\ &\cong \text{Hom}^{m-n-1}(\mathbb{1}_{n+2}, \bigoplus_{[n+2]} \mathbb{1}_{n+2}) \end{aligned}$$

where the last line follows since $m < 2(n+2)$ meaning $\text{Hom}^m(\mathbf{E}\mathbb{1}_{n+2}, \mathbf{E}\mathbb{1}_{n+2} \langle -2(n+2) \rangle) = 0$. Note that in the third isomorphism above we use (3.10). Keeping track of degrees, we find that

$$(3.14) \quad \text{Hom}^m(\mathbf{E}\mathbb{1}_n, \mathbf{E}\mathbb{1}_n) \cong \bigoplus_{k \geq 0} \text{Hom}(\mathbb{1}_{n+2}, \mathbb{1}_{n+2} \langle m-2k \rangle).$$

If $f \in \text{Hom}^m(\mathbf{E}\mathbb{1}_n, \mathbf{E}\mathbb{1}_n)$ then we denote the map induced by adjunction $f' \in \text{Hom}^m(\mathbb{1}_{n+2}, \mathbf{E}\mathbb{F}\mathbb{1}_{n+2} \langle -(n+1) \rangle)$ and the induced maps on the right side of (3.14) by $f_k \in \text{Hom}(\mathbb{1}_{n+2}, \mathbb{1}_{n+2} \langle m-2k \rangle)$.

Now let us trace through the series of isomorphisms above in order to explicitly identify f' with f_k . The critical isomorphism is the third one where one uses the isomorphism

$$\mathbf{F}\mathbf{E}\mathbb{1}_{n+2} \bigoplus_{k=0}^{n+1} \mathbb{1}_{n+2} \langle -n-1+2k \rangle \rightarrow \mathbf{E}\mathbb{F}\mathbb{1}_{n+2}$$

from Corollary 3.11. Thus we find that f' corresponds to the composition

$$\mathbb{1}_{n+2} \xrightarrow{f_k} \mathbb{1}_{n+2}\langle m-2k \rangle \xrightarrow{\text{cup}} \text{EF}\mathbb{1}_{n+2}\langle m-(n+1)-2k \rangle \xrightarrow{k \uparrow \downarrow} \text{EF}\mathbb{1}_{n+2}\langle m-(n+1) \rangle.$$

Consequently, using the adjunction which relates f and f' we find that f is given by the composition

$$\mathbb{1}_{n+2}\text{E} \xrightarrow{f_k \uparrow} \mathbb{1}_{n+2}\text{E}\langle m-2k \rangle \xrightarrow{k \uparrow} \mathbb{1}_{n+2}\text{E}\langle m \rangle.$$

This implies what we needed to prove.

If $m = 2$ and $n = -1$ then the long calculation above yields

$$\text{Hom}^2(\text{E}\mathbb{1}_{-1}, \text{E}\mathbb{1}_{-1}) \cong \text{Hom}^2(\mathbb{1}_1, \mathbb{1}_1) \oplus \text{Hom}^2(\text{E}\mathbb{1}_1, \text{E}\mathbb{1}_1).$$

The space of maps on the right is one-dimensional and is induced by the dot. The result follows. \square

4. CYCLIC BIADJOINTNESS

In this section we fix all adjunction maps and prove cyclic biadjointness when $\mathfrak{g} = \mathfrak{sl}_2$. This means that the two possible ways for defining dots on downward strands, sideways crossings or downward crossings give the same 2-morphisms.

4.1. Fixing adjunction maps. Recall that at this point, adjunction maps are only determined up to a scalar. In this section we rescale them so that caps and cups are adjoint to each other and so that

$$(4.1) \quad \begin{array}{c} n \\ \circlearrowleft \\ n-1 \end{array} = \text{Id}_{\mathbb{1}_n} \quad \text{for } n \geq 1, \quad \begin{array}{c} n \\ \circlearrowright \\ -n-1 \end{array} = \text{Id}_{\mathbb{1}_n} \quad \text{for } n < -1.$$

Recall that negative degree dotted bubbles are zero while a dotted bubble of degree zero must be a non-zero multiple of the identity map by Corollary 3.6 (note that for $n = 0$ there are no degree zero dotted bubbles). By rescaling the adjunction maps we can ensure the degree zero bubbles satisfy the conditions above. More precisely, we rescale in the following order.

$m \geq 0$				
	fixed arbitrarily	determined by adjunction	fixed by value of bubble	determined by adjunction
$m < 0$				
	fixed arbitrarily	determined by adjunction	fixed by value of bubble	determined by adjunction

The only time this rescaling fails is when $m = 0$ above. In that case, the rescaling for $m \geq 0$ fixes the value of the positive bubble in weight $n = 1$, so that

$$(4.2) \quad \begin{array}{c} +1 \\ \circlearrowleft \end{array} = \text{Id}_{\mathbb{1}_n}$$

where $+1$ denotes the outside region of the bubble. However, the rescaling for $m = 0$ fails to rescale the positive degree bubble in region $n = -1$. This bubble is multiplication by some arbitrary scalar c_{-1} :

$$(4.3) \quad \overset{-1}{\bigcirc} = c_{-1} \text{Id}_{\mathbb{1}_n}$$

where -1 denotes the outside region of the bubble. We will show in Lemma 5.3 that in fact $c_{-1} = 1$.

4.2. Cyclic biadjointness. Recall that the action of dots and crossings is originally defined only for upward pointing strands. To define dots on downward strands we use the (bi)adjunction between \mathbf{E} and \mathbf{F} . We define

$$(4.4) \quad \begin{array}{c} n \\ \downarrow \\ \bullet \\ \downarrow \\ n+2 \end{array} := \begin{array}{c} \text{strand } n+2 \text{ with dot} \\ \text{strand } n \end{array} \quad \text{and} \quad \begin{array}{c} n \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ n \end{array} := \begin{array}{c} \text{strand } n \text{ with dot} \\ \text{strand } n+2 \end{array}$$

However, there is another way to use adjunction to define these maps. The next two results (Lemmas 4.1 and 4.2) show that both ways give the same map (we refer to this as the “cyclic biadjointness property”).

Lemma 4.1. *Given a strong 2-representation of \mathfrak{sl}_2 we have:*

$$(4.5) \quad \begin{array}{c} \text{strand } n+2 \text{ with dot} \\ \text{strand } n \end{array} = \begin{array}{c} \text{strand } n \text{ with dot} \\ \text{strand } n+2 \end{array}$$

Proof. The analogue of Lemma 3.12 for downward pointing strands (obtained by taking adjoints) implies that for $n \geq -1$ we have

$$(4.6) \quad \begin{array}{c} \text{strand } n+2 \text{ with dot} \\ \text{strand } n \end{array} = \gamma_0 \begin{array}{c} \text{strand } n \text{ with dot} \\ \text{strand } n+2 \end{array} + \begin{array}{c} \text{strand } n \\ \text{strand } n+2 \end{array} \boxed{\gamma_1}$$

for $\gamma_0 \in \text{End}(\mathbb{1}_{n+2}) = \mathbb{k}$ and $\gamma_1 \in \text{End}^2(\mathbb{1}_{n+2})$. Closing off this relation on the left with m dots gives rise to the equality

$$(4.7) \quad \begin{array}{c} m \text{ dots} \\ \text{strand } n+2 \text{ with dot} \\ \text{strand } n \end{array} = \gamma_0 \begin{array}{c} m \text{ dots} \\ \text{strand } n \text{ with dot} \\ \text{strand } n+2 \end{array} + \begin{array}{c} m \text{ dots} \\ \text{strand } n \\ \text{strand } n+2 \end{array} \boxed{\gamma_1}$$

which simplifies to

$$(4.8) \quad (1 - \gamma_0) \begin{array}{c} n+2 \\ \bigcirc \\ m+1 \end{array} = \begin{array}{c} n+2 \\ \bigcirc \\ m \end{array} \boxed{\gamma_1}$$

For $n \geq 0$ take $m = n$ so that the right-hand side is zero since the dotted bubble has negative degree. Hence $\gamma_0 = 1$. Taking $m = n + 1$ then implies that $\gamma_1 = 0$. A similar argument proves the lemma for $n < -1$.

The case $n = -1$ requires special attention. Note that for $n = -1$ the diagram

$$(4.9) \quad \begin{array}{c} \text{Diagram: A vertical line with a loop on the right side, labeled } n. \\ \text{The loop is formed by two strands crossing, with the right strand looping back to the left.} \end{array}$$

is zero since it has degree -2 . Using the NilHecke relation we have

$$(4.10) \quad \begin{array}{c} \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ = \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ - r_i \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ \stackrel{(4.2)}{=} - r_i \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ = \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ - r_i \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ = \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \end{array}$$

so that these dotted curls are non-zero. Hence, gluing equation (4.6) into the diagram

$$\begin{array}{c} \text{Diagram: A vertical line with a loop on the right side, labeled } n+2. \\ \text{The loop is formed by two strands crossing, with the right strand looping back to the left.} \end{array}$$

and simplifying using the adjoint structure implies that $\gamma_0 = 1$. Then equation (4.8) with $m = 0$ implies $\gamma_1 = 0$. \square

Lemma 4.2. *In a strong 2-representation of \mathfrak{sl}_2 we have:*

$$(4.11) \quad \begin{array}{c} \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ = \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \end{array}$$

Proof. We must have

$$(4.12) \quad \begin{array}{c} \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ = \kappa \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \end{array}$$

for some scalar κ . Using cyclicity for dots proven above, this equation implies

$$(4.13) \quad \begin{array}{c} \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \\ = \kappa \text{Diagram: A vertical line with a loop on the right side, labeled } n, with a dot on the left strand.} \end{array}$$

Then by the NilHecke relation we have

$$(4.14) \quad \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + r_i \begin{array}{c} \text{Diagram 3} \end{array}$$

$$(4.15) \quad \kappa \begin{array}{c} \text{Diagram 4} \end{array} = \kappa \begin{array}{c} \text{Diagram 5} \end{array} + r_i \kappa \begin{array}{c} \text{Diagram 6} \end{array}$$

Using the equality (4.14), the cyclic condition for the dot 2-morphism, together with the biadjoint relations implies

$$(4.16) \quad \begin{array}{c} \text{Diagram 7} \end{array} = \kappa \begin{array}{c} \text{Diagram 8} \end{array}$$

so that $\kappa = 1$. □

The sideways crossings were defined up to a scalar by Lemma 3.10. We now fix these scalars as follows.

$$(4.17) \quad \begin{array}{c} \text{Diagram 9} \end{array} := \begin{array}{c} \text{Diagram 10} \end{array} \quad \begin{array}{c} \text{Diagram 11} \end{array} := \begin{array}{c} \text{Diagram 12} \end{array}$$

5. PROOF OF THE EXTENDED \mathfrak{sl}_2 RELATIONS

By the remarks in section 2.1.3, if $I = \{i\}$ then by rescaling degrees it suffices to work with Cartan datum associated to \mathfrak{sl}_2 .

5.1. Symmetric functions and dotted bubbles. Let Sym denote the graded ring of symmetric functions in countably infinite variables. As a graded vector space $\text{Sym} = \bigoplus_{\ell=0}^{\infty} \text{Sym}^{\ell}$ where Sym^{ℓ} denote the homogeneous symmetric functions of degree ℓ . The ring Sym can be identified with the polynomial ring $\mathbb{Z}[e_1, e_2, \dots]$ and the polynomial ring $\mathbb{Z}[h_1, h_2, \dots]$ where e_j is the j th elementary symmetric function, and h_j is the j th complete symmetric function. These symmetric functions are related by the equation

$$(5.1) \quad \sum_{\ell_1 + \ell_2 = m} (-1)^{\ell_2} e_{\ell_1} h_{\ell_2} = \delta_{m,0},$$

where $e_0 = h_0 = 1$ and $e_r = h_r = 0$ for $r < 0$. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ let $e_{\lambda} = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_m}$.

From equation (5.1) one can recursively rewrite the complete symmetric functions in terms of the elementary symmetric functions

$$(5.2) \quad h_r = \sum_{\lambda: |\lambda|=r} \alpha_{\lambda} e_{\lambda}$$

for some coefficients $\alpha_{\lambda} \in \mathbb{Z}$.

Define a grading preserving map

$$(5.3) \quad \varphi^n: \text{Sym} \longrightarrow \text{End}(\mathbb{1}_n)$$

$$(5.4) \quad e_\lambda = e_{\lambda_1} \dots e_{\lambda_m} \mapsto \begin{cases} \begin{array}{c} n \\ \text{bubble}_{n-1+\lambda_1} \dots \text{bubble}_{n-1+\lambda_m} \end{array} & \text{if } n > 0 \\ \begin{array}{c} n \\ \text{bubble}_{-n-1+\lambda_1} \dots \text{bubble}_{-n-1+\lambda_m} \end{array} & \text{if } n < 0. \end{cases}$$

and write

$$(5.5) \quad e_{\lambda,n} := \varphi^n(e_\lambda),$$

with $\deg(e_{\lambda,n}) = 2(\lambda_1 + \lambda_2 + \dots + \lambda_m)$. When λ is the empty partition we write $e_{\lambda,n} = e_\emptyset = 1$.

It was shown in [17, Proposition 8.2] that this map is an isomorphism in the 2-category $\mathcal{U}(\mathfrak{sl}_2)$. In this context, the map φ^n provides a natural interpretation to fake bubbles introduced in section 2.6. Indeed, comparing the homogeneous terms 2.18 with equation (5.1) of the infinite Grassmannian equation show that the degree r fake bubble in region n corresponds to $\varphi^n((-1)^r h_r)$. In particular, in the 2-category $\mathcal{U}(\mathfrak{sl}_2)$ one has

$$(5.6) \quad \begin{array}{c} n \\ \text{bubble}_{n-1+r} \end{array} = \begin{cases} \sum_{\lambda: |\lambda|=r} \alpha_\lambda \begin{array}{c} n \\ \text{bubble}_{-n-1+\lambda_1} \dots \text{bubble}_{-n-1+\lambda_m} \end{array} & \text{if } 0 \leq r < -n+1 \\ 0 & r < 0. \end{cases}$$

for $n \leq 0$, and

$$(5.7) \quad \begin{array}{c} n \\ \text{bubble}_{-n-1+r} \end{array} = \begin{cases} \sum_{\lambda: |\lambda|=r} \alpha_\lambda \begin{array}{c} n \\ \text{bubble}_{n-1+\lambda_1} \dots \text{bubble}_{n-1+\lambda_m} \end{array} & \text{if } 0 \leq r < n+1 \\ 0 & r < 0. \end{cases}$$

for $n \geq 0$, where for $n = 0$ we define

$$(5.8) \quad \begin{array}{c} 0 \\ \text{bubble}_{-1} \end{array} = \text{Id}_{\mathbb{1}_0}, \quad \begin{array}{c} 0 \\ \text{bubble}_{-1} \end{array} = \text{Id}_{\mathbb{1}_0}.$$

See [19, Section 3.6] for more details.

5.2. A general form of the \mathfrak{sl}_2 commutator relation. All string diagrams in this subsection are interpreted in \mathcal{K} , for some 2-category \mathcal{K} admitting a strong 2-representation of \mathfrak{sl}_2 . We also take $n \geq 0$ throughout this section. Results for $n \leq 0$ are proven similarly.

By Corollary 3.11 the map

$$\zeta := \bigotimes_{k=0}^{n-1} \text{bubble}_{k-1} : \text{FE } \mathbb{1}_n \bigoplus_{k=0}^{n-1} \mathbb{1}_n \langle n-1-2k \rangle \rightarrow \text{EF } \mathbb{1}_n$$

is invertible. We describe its inverse ζ^{-1} diagrammatically as follows:

$$(5.9) \quad n \quad \boxed{\zeta(n)}^n = \bigoplus_{k=0}^{n-1} \left(\boxed{\zeta(n-1-k)}^n \right) : \text{EF} \mathbb{1}_n \rightarrow \text{FE} \mathbb{1}_n \bigoplus_{k=0}^{n-1} \mathbb{1}_n \langle n-1-2k \rangle.$$

Condition 3 of Definition 1.1 only requires the *existence* of isomorphisms between the two 1-morphisms on either side. However, the space of 2-morphisms between a pair of 1-morphisms in \mathcal{K} could contain maps that cannot be expressed using 2-morphisms from the strong 2-representation of \mathfrak{sl}_2 , i.e. using dots, crossings, caps and cups. In the next proposition we show that this is not the case for the 2-morphisms giving the isomorphism ζ^{-1} .

Proposition 5.1. *The isomorphism ζ^{-1} has the form*

$$(5.10) \quad n \quad \boxed{\zeta(n)}^n = \beta_n \quad \text{crossing}^n \quad \boxed{\zeta(\ell)}^n = \sum_{|\lambda|+j=\ell} \alpha_\lambda^\ell(n) e_{\lambda,n}^j \quad \text{dotted bubble}^n$$

for some coefficients $\beta_n \in \mathbb{k}^\times$ and $\alpha_\lambda^\ell(n) \in \mathbb{k}$.

Proof. The first claim in the Proposition follows immediately from Lemma 3.10. For the second claim take adjoints in Lemma 3.12 equation (3.12) so that

$$(5.11) \quad \boxed{\zeta(\ell)}^n = \sum_{i=0}^{\ell} \boxed{f_i(\ell)}^n : \text{EF} \mathbb{1}_n \rightarrow \mathbb{1}_n \langle 2\ell+1-n \rangle$$

for some 2-endomorphisms $f_i(\ell) \in \text{End}_{\mathcal{U}}(\mathbb{1}_n, \mathbb{1}_n \langle 2i \rangle)$.

The component of $\zeta^{-1}\zeta$ mapping the summand $\mathbb{1}_n \langle 2\ell'+1-n \rangle$ to the summand $\mathbb{1}_n \langle 2\ell+1-n \rangle$ is given by the composite

$$(5.12) \quad \mathbb{1}_n \langle 2\ell'+1-n \rangle \xrightarrow{\text{cup}} \mathbb{1}_n \xrightarrow{\sum_i \boxed{f_i(\ell)}^n} \mathbb{1}_n \langle 2\ell+1-n \rangle.$$

The condition $\zeta^{-1}\zeta = \text{Id}$ implies that this composite must satisfy the equation

$$(5.13) \quad \sum_{i=0}^{\ell} \boxed{f_i(\ell)}^n = \delta_{\ell,\ell'} \quad \text{cap}^n$$

for $0 \leq \ell', \ell \leq n-1$.

By varying ℓ' for fixed ℓ , it is possible to rewrite each of the 2-endomorphisms $f_i(\ell)$ as products of dotted bubbles. For example, by setting $\ell' = \ell$ it follows that the degree zero 2-endomorphism $f_0(\ell)$ is multiplication by the scalar 1. Continuing by induction, decreasing ℓ' shows that all the $f_i(\ell)$ can be rewritten as a linear combination

$$(5.14) \quad \boxed{f_i(\ell)}^n = \sum_{\lambda: |\lambda|=i} \alpha_\lambda^\ell(n) e_{\lambda,n}$$

of products $e_{\lambda,n}$ of dotted bubbles for some $\alpha_\lambda^\ell(n) \in \mathbb{k}$, completing the proof. \square

For notational simplicity we write $\alpha_\lambda^\ell(n)$ for all ℓ and λ and n , but set $\alpha_\lambda^\ell(n) = 0$ unless $|\lambda| \leq \ell \leq n-1$.

5.3. Relations resulting from the \mathfrak{sl}_2 commutator relation. The plan in this section is to show that $\alpha_\lambda^{(\ell)}(n)$ is equal to α_λ (as defined in equation 5.2) and moreover that $\beta_n = -r_i^2$. Observe that ζ^{-1} in Proposition 5.1 is the inverse of ζ if and only if the following relations hold in \mathcal{K} :

Relations for $n \geq 0$	
(A1)	$\mathbb{1}_n \langle 2\ell' + 1 - n \rangle \rightarrow \mathbb{1}_n \langle 2\ell + 1 - n \rangle$ $\delta_{\ell, \ell'} = \sum_{\lambda: \lambda \leq \ell} \alpha_\lambda^\ell(n) e_{\lambda, n} \text{ (diagram: a circle with a dot at the bottom, labeled } n \text{ at the top and } n-1+\ell-\ell'- \lambda \text{ at the bottom)}$
(A2)	$\text{FE} \mathbb{1}_n \rightarrow \text{FE} \mathbb{1}_n$ $n \text{ (two parallel vertical lines with arrows pointing down)} = \beta_n \text{ (diagram: two crossing lines with arrows pointing down, labeled } n \text{ at the top right)}$
(A3)	$\mathbb{1}_n \langle 2\ell' + 1 - n \rangle \rightarrow \text{FE} \mathbb{1}_n$ $\beta_n \text{ (diagram: a loop with a dot at the bottom, labeled } n-1-\ell \text{ at the bottom)} = 0$
(A4)	$\text{FE} \mathbb{1}_n \rightarrow \mathbb{1}_n \langle 2\ell + 1 - n \rangle$ $\sum_\lambda \alpha_\lambda^\ell(n) e_{\lambda, n} \text{ (diagram: a loop with a dot at the bottom, labeled } n-1-\ell \text{ at the top)} = 0$
(A5)	$\text{EF} \mathbb{1}_n \rightarrow \text{EF} \mathbb{1}_n$ $n \text{ (two parallel vertical lines with arrows pointing down)} = \beta_n \text{ (diagram: two crossing lines with arrows pointing down, labeled } n \text{ at the top right)} + \sum_{\substack{f_1+f_2+ \lambda \\ = n-1}} \alpha_\lambda^{ \lambda +f_2}(n) e_{\lambda, n} \text{ (diagram: two arcs, one labeled } n \text{ at the top and } f_1 \text{ at the bottom right, the other labeled } f_2 \text{ at the bottom left)}$

Proposition 5.2. *Setting the coefficients $\alpha_\lambda^\ell(n)$ from Proposition 5.1 equal to the coefficients α_λ from equation (5.2) gives a solution to equations (A1)–(A5).*

Proof. All the curls in the summation (A4) are zero because of (A3) (note that you can rotate the curl in (A3) by using adjunction). So relation (A4) follows immediately from (A3) and does not impose any conditions on the coefficients $\alpha_\lambda^\ell(n)$.

The only other equations that involve the parameters $\alpha_\lambda^\ell(n)$ are equations (A1) and (A5). First we note that equations (A1) and (A5) can be written in terms of

$$h_s^\ell := (-1)^s \sum_{\lambda: |\lambda|=s} \alpha_\lambda^\ell(n) e_{\lambda, n} \quad \text{for } n \in \mathbb{Z},$$

where $0 \leq s \leq \ell$. More specifically, (A1) can be written as

$$(5.15) \quad \delta_{b,0} = \sum_{\lambda: |\lambda| \leq b} \alpha_\lambda^\ell(n) e_{\lambda,n} e_{b-|\lambda|,n} \iff \delta_{b,0} = \sum_{s \leq b} (-1)^s h_s^\ell e_{b-s,n}$$

where $b = \ell - \ell'$ while (A5) can be written as

$$(5.16) \quad \begin{array}{c} n \\ \downarrow \\ \downarrow \\ n \end{array} = \beta_n \begin{array}{c} \text{X} \\ \downarrow \\ \downarrow \\ n \end{array} + \sum_{f_1+f_2 < n} (-1)^{n-1-f_1-f_2} h_{n-1-f_1-f_2}^{n-1-f_1} \begin{array}{c} n \\ \text{cap} \\ f_1 \\ \text{cup} \\ f_2 \end{array}$$

Now, from (5.15) we can solve uniquely for h_b^ℓ in terms of h_s^ℓ for $s < b$. So we just need to choose $\alpha_\lambda^\ell(n)$ so that (5.15) is satisfied. However we can rewrite (5.15) as

$$\sum_{s+r=b} (-1)^s h_s^\ell e_r = \delta_{b,0}$$

where $e_r = e_{r,n}(n)$. This means that if we choose $\alpha_\lambda^\ell(n) = \alpha_\lambda$ then (A1) is satisfied and we are done. \square

Since, by the above proposition, the coefficients $\alpha_\lambda^\ell(n)$ are independent of ℓ and n we will just write $\alpha_\lambda = \alpha_\lambda^{|\lambda|}(n) = \alpha_\lambda^\ell(n)$ for all n and $0 \leq \ell \leq n-1$.

It is clear that relations (A1)–(A5) allow you to simplify clockwise oriented curls. Less obvious is that they also allow you to simplify counter-clockwise curls. To see this add a cap with n dots to the bottom of equation (A5) and simplifying using the NilHecke relations and equation (A3), then use equation (A1) noting that $\alpha_0^0(n) = 1$. This way you arrive at a formula for simplifying any type of curl.

We now summarize this fact together with relations (A1)–(A5) in a more compact form utilizing fake bubbles in \mathcal{K} defined as in (5.6) and (5.7). We write down just the case $n \geq 0$ as the case $n < 0$ is similar.

- The biadjointness condition, the cyclic condition, and the NilHecke algebra axioms.
- Negative degree bubbles are still zero, but a dotted bubble of degree zero is multiplication by 1 for $n \neq -1$, and equal to multiplication by c_{-1} when $n = -1$.
- If $n > 0$ the following relations hold:

$$(5.17) \quad \begin{array}{c} \text{X} \\ \downarrow \\ \downarrow \\ n \end{array} = 0 \quad \begin{array}{c} n \\ \text{X} \\ \downarrow \\ \downarrow \end{array} = -\frac{1}{r_i \beta_n} \sum_{g_1+g_2=n} \begin{array}{c} n \\ \text{cap} \\ -n-1+g_2 \end{array} \begin{array}{c} \bullet \\ g_1 \\ \downarrow \end{array}$$

$$(5.18) \quad \begin{array}{c} \downarrow \\ \downarrow \\ n \end{array} = \beta_n \begin{array}{c} \text{X} \\ \downarrow \\ \downarrow \\ n \end{array} + \sum_{\substack{f_1+f_2+f_3 \\ =n-1}} \begin{array}{c} n \\ \text{cap} \\ f_1 \\ \text{cup} \\ f_2 \\ \text{cup} \\ f_3 \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ n \end{array} = \beta_n \begin{array}{c} \text{X} \\ \downarrow \\ \downarrow \\ n \end{array}$$

- If $n = 0$ then

$$(5.19) \quad \begin{array}{c} \downarrow \\ \downarrow \\ n \end{array} = \beta_0 \begin{array}{c} \text{X} \\ \downarrow \\ \downarrow \\ n \end{array} \quad \begin{array}{c} \downarrow \\ \downarrow \\ n \end{array} = \beta_0 \begin{array}{c} \text{X} \\ \downarrow \\ \downarrow \\ n \end{array}$$

Also, we have that

$$(5.20) \quad \deg \left(\begin{array}{c} \text{X} \\ n \end{array} \right) = 0, \quad \deg \left(\begin{array}{c} n \\ \text{X} \end{array} \right) = 0.$$

Since the space of degree zero endomorphisms $\mathbf{E}\mathbb{1}_n$ is one-dimensional in \mathcal{K} , we must have that both of the above morphisms are multiples of the identity 2-morphism $\text{Id}_{\mathbf{E}\mathbb{1}_n}$. We keep track of these multiples as follows:

$$(5.21) \quad \begin{array}{c} \text{X} \\ n \end{array} = -c_0^+ \begin{array}{c} | \\ n \end{array}, \quad \begin{array}{c} n \\ \text{X} \end{array} = c_0^- \begin{array}{c} | \\ n \end{array}$$

Lemma 5.3.

- The coefficient c_{-1} from equation (4.3) satisfies $c_{-1} = 1$.
- For all values of n such that $\mathbb{1}_n$ is non-zero, we have $\beta_n = -r_i^2$.
- The coefficients introduced in equation (5.21) satisfy $c_0^+ = c_0^- = r_i$.

Proof. The first equality follows from the equalities

$$(5.22) \quad 0 = \begin{array}{c} +1 \\ \text{X} \end{array} = \begin{array}{c} -1 \\ \text{X} \end{array} \stackrel{(5.18)}{=} \frac{1}{\beta_1} \begin{array}{c} | \\ -1 \end{array} - \frac{1}{\beta_1} \begin{array}{c} \text{X} \\ -2 \end{array}$$

where the second equality follow from the $n \leq 0$ analog of equation 5.18.

Using the definition of fake bubbles this implies

$$(5.23) \quad 0 = \frac{1}{\beta_1} \left(\begin{array}{c} | \\ c_{-1} \end{array} - \begin{array}{c} | \\ -1 \end{array} \right)$$

so that $c_{-1} = 1$.

For $n \geq 0$ it follows that

$$(5.24) \quad \frac{1}{\beta_n} \begin{array}{c} n+2 \\ \text{X} \\ (n+2)^{-1} \end{array} \stackrel{(5.18)}{=} \begin{array}{c} n+2 \\ \text{X} \\ n+1 \end{array} = \begin{array}{c} n \\ \text{X} \\ n+1 \end{array} \stackrel{(2.15)}{=} -r_i \sum_{\ell_1 + \ell_2 = n} \begin{array}{c} \ell_2 \\ \text{X} \\ \ell_1 \end{array} = -r_i \begin{array}{c} n \\ \text{X} \\ n \end{array}$$

When $n = 0$ the above together with equation (5.21) implies $c_0^+ = -\frac{1}{\beta_0 r_i}$. For $n > 0$, simplifying using the NilHecke dot slide formula implies $\beta_n = -r_i^{-2}$. A similar calculation for $n \leq 0$ using a clockwise oriented bubble with $-n + 1$ dots implies $\beta_n = -r_i^{-2}$ for $n < 0$ and $c_0^- = \frac{1}{\beta_0 r_i}$.

Capping off the top of (5.19) for $n = 0$ and simplifying the resulting diagram implies that $\beta_0 = -r_i^2$, and that $c_0^+ = c_0^- = r_i$. \square

Theorem 5.4. Any strong 2-representation of \mathfrak{sl}_2 on a 2-category \mathcal{K} extends to a 2-representation $\mathcal{U}_Q(\mathfrak{g}) \rightarrow \mathcal{K}$.

Proof. This proof is immediate from Lemma 5.3 and relations (5.17) through (5.19). \square

6. PROOF OF THE NON- \mathfrak{sl}_2 RELATIONS

Fix a given Q -strong 2-representation of \mathfrak{g} . In this section we prove the relations in $\mathcal{U}_Q(\mathfrak{g})$ involving strands with different labels (namely the mixed relations and Q -cyclicity involving differently labeled strands). We refer to these as non- \mathfrak{sl}_2 relations.

6.1. Preliminary results. We begin with a couple of results on spaces of Homs.

Lemma 6.1. *For $i \neq j \in I$, we have*

$$(6.1) \quad \text{Hom}^k(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_j \mathbf{E}_i \mathbb{1}_\lambda) \cong \begin{cases} 0 & \text{if } k < -(\alpha_i, \alpha_j) \\ \mathbb{k} & \text{if } k = -(\alpha_i, \alpha_j) \end{cases}$$

while

$$(6.2) \quad \text{Ext}^k(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda) \cong \begin{cases} 0 & \text{if } k < 0, \\ \mathbb{k} & \text{if } k = 0. \end{cases}$$

The same results hold if we replace all the \mathbf{E} s by \mathbf{F} s.

Proof. If the Dynkin diagram of \mathfrak{g} is simply laced then this result follows from Lemma 4.5 of [4]. But since we are dealing with arbitrary symmetrizable Kac-Moody algebras we reproduce the proof here.

Suppose $\langle j, \lambda \rangle \leq 0$ (the case $\langle j, \lambda \rangle \geq 0$ is similar). The proof is then by induction on $\langle j, \lambda \rangle$. The base case is vacuous since $\mathbb{1}_\lambda = 0$ if $\langle j, \lambda \rangle \ll 0$. For notational simplicity we will denote $d_i := (\alpha_i, \alpha_i)/2$ for any $i \in I$. Note that under these conventions we have

$$(\mathbf{E}_i \mathbb{1}_\lambda)_L \cong \mathbb{1}_\lambda \mathbf{F}_i \langle -(\lambda, \alpha_i) - d_i \rangle \quad \text{and} \quad (\mathbb{1}_\lambda \mathbf{F}_i)_L \cong \mathbf{E}_i \mathbb{1}_\lambda \langle (\lambda, \alpha_i) + d_i \rangle.$$

Now:

$$\begin{aligned} & \text{Hom}^k(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_j \mathbf{E}_i \mathbb{1}_\lambda) \\ & \cong \text{Hom}^k((\mathbf{E}_j \mathbb{1}_{\lambda + \alpha_i})_L \mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_i \mathbb{1}_\lambda) \\ & \cong \text{Hom}^k(\mathbf{F}_j \langle -(\lambda + \alpha_i, \alpha_j) - d_j \rangle \mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_i \mathbb{1}_\lambda) \\ & \cong \text{Hom}^k(\mathbf{E}_i \mathbf{F}_j \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_i \langle (\lambda + \alpha_i, \alpha_j) + d_j \rangle \mathbb{1}_\lambda) \\ & \cong \text{Hom}^k(\oplus_{[-\langle \lambda, j \rangle]_j} \mathbf{E}_i \mathbb{1}_\lambda, \mathbf{E}_i \langle (\lambda + \alpha_i, \alpha_j) + d_j \rangle \mathbb{1}_\lambda) \oplus \text{Hom}^k(\mathbf{E}_i \mathbf{E}_j \mathbf{F}_j \mathbb{1}_\lambda, \mathbf{E}_i \langle (\lambda + \alpha_i, \alpha_j) + d_j \rangle \mathbb{1}_\lambda) \\ & \cong \bigoplus_{s=0}^{-\langle j, \lambda \rangle - d_j} \text{Hom}^k(\mathbf{E}_i \mathbb{1}_\lambda, \mathbf{E}_i \langle (\alpha_i, \alpha_j) - 2sd_j \rangle \mathbb{1}_\lambda) \\ & \quad \oplus \text{Hom}^k(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_{\lambda - \alpha_j}, \mathbf{E}_i (\mathbb{1}_{\lambda - \alpha_j} \mathbf{F}_j)_L \langle (\lambda + \alpha_i, \alpha_j) + d_j \rangle) \end{aligned}$$

where to get the last equality we use that $\langle j, \lambda \rangle \cdot d_j = (\alpha_j, \lambda)$. Now, $\text{Hom}^k(\mathbf{E}_i \mathbb{1}_\lambda, \mathbf{E}_i \mathbb{1}_\lambda)$ is zero if $k < 0$ and equals \mathbb{k} if $k = 0$. So the summation above is zero if $k + (\alpha_i, \alpha_j) < 0$ and \mathbb{k} if $k + (\alpha_i, \alpha_j) = 0$. Meanwhile the second term above equals

$$(6.3) \quad \text{Hom}^k(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_{\lambda - \alpha_j}, \mathbf{E}_i \mathbf{E}_j \langle 2(\lambda, \alpha_j) + (\alpha_i, \alpha_j) \rangle \mathbb{1}_{\lambda - \alpha_j}).$$

Using adjunction again together with induction one can show that $\text{Hom}^t(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_{\lambda - \alpha_j}, \mathbf{E}_i \mathbf{E}_j \mathbb{1}_{\lambda - \alpha_j})$ is zero if $t < 0$ and \mathbb{k} if $t = 0$. Thus (6.3) vanishes since $\langle \alpha_i, \alpha_j \rangle < 0$ and this completes the proof of (6.1). The proof of equation (6.2) is similar. \square

Corollary 6.2. *For $i \neq j \in I$, we have*

$$(6.4) \quad \text{Hom}^k(\mathbf{E}_i \mathbf{F}_j \mathbb{1}_\lambda, \mathbf{F}_j \mathbf{E}_i \mathbb{1}_\lambda) \cong \begin{cases} 0 & \text{if } k < 0 \\ \mathbb{k} & \text{if } k = 0 \end{cases}$$

The same results hold if we interchange all the \mathbf{E} s with \mathbf{F} s and all \mathbf{F} s with \mathbf{E} s.

Proof. By adjunction we have:

$$\begin{aligned}
 & \text{Hom}^k(\mathbf{E}_i \mathbf{F}_j \mathbb{1}_\lambda, \mathbf{F}_j \mathbf{E}_i \mathbb{1}_\lambda) \\
 & \cong \text{Hom}^k((\mathbb{1}_{\lambda+\alpha_i-\alpha_j} \mathbf{F}_j)_L \mathbf{E}_i \mathbb{1}_{\lambda-\alpha_j}, \mathbf{E}_i (\mathbb{1}_{\lambda-\alpha_j} \mathbf{F}_j)_L \mathbb{1}_{\lambda-\alpha_j}) \\
 & \cong \text{Hom}^k(\mathbf{E}_j \mathbf{E}_i \langle (\lambda + \alpha_i - \alpha_j, \alpha_j) + d_j \rangle \mathbb{1}_{\lambda-\alpha_j}, \mathbf{E}_i \mathbf{E}_j \langle (\lambda - \alpha_j, \alpha_j) + d_j \rangle \mathbb{1}_{\lambda-\alpha_j}) \\
 & \cong \text{Hom}^k(\mathbf{E}_j \mathbf{E}_i \mathbb{1}_{\lambda-\alpha_j}, \mathbf{E}_i \mathbf{E}_j \langle -(\alpha_i, \alpha_j) \rangle \mathbb{1}_{\lambda-\alpha_j})
 \end{aligned}$$

where, as before, $d_j = (\alpha_j, \alpha_j)/2$. By lemma 6.1, this is isomorphic to zero if $k < 0$ and \mathbb{k} if $k = 0$. \square

6.2. Mixed relations $\mathbf{E}_i \mathbf{F}_j \cong \mathbf{F}_j \mathbf{E}_i$. Recall the definitions of sideways crossings from (2.6) and (2.7). Note that these 2-morphisms are non-zero since they are related by adjunction to the unique maps in $\text{Hom}^{-(\alpha_i, \alpha_j)}(\mathbf{E}_i \mathbf{E}_j \mathbb{1}_\lambda, \mathbf{E}_j \mathbf{E}_i \mathbb{1}_\lambda)$.

Composing these maps gives rise to degree zero endomorphisms of $\mathbf{E}_i \mathbf{F}_j \mathbb{1}_\lambda$ and of $\mathbf{F}_i \mathbf{E}_j \mathbb{1}_\lambda$ for $i \neq j$. Since $\mathbf{E}_i \mathbf{F}_j \mathbb{1}_\lambda \cong \mathbf{F}_j \mathbf{E}_i \mathbb{1}_\lambda$ these Hom spaces are 1-dimensional in degree zero by corollary 6.2. Thus we must have

$$(6.5) \quad \begin{array}{c} \text{sideways crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array} := \begin{array}{c} \text{curly crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array} = \gamma_{ij}(\lambda) \begin{array}{c} \text{vertical crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array}$$

$$(6.6) \quad \begin{array}{c} \text{sideways crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array} := \begin{array}{c} \text{curly crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array} = \beta_{ij}(\lambda) \begin{array}{c} \text{vertical crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array}$$

for some $\gamma_{ij}(\lambda), \beta_{ij}(\lambda) \in \mathbb{k}$.

Proposition 6.3. *For all $i, j \in I$ with $i \neq j$ the coefficients from (6.5) and (6.6) satisfy.*

$$\beta_{ij}(\lambda) = \gamma_{ji}(\lambda) = t_{ij}.$$

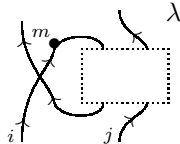
Proof. Equations (6.5) and (6.6) imply

$$(6.7) \quad \gamma_{ji}(\lambda) \begin{array}{c} \text{vertical crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array} = \begin{array}{c} \text{sideways crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array} = \beta_{ij}(\lambda) \begin{array}{c} \text{vertical crossing} \end{array} \begin{array}{c} \lambda \\ i \quad j \end{array}$$

so that $\gamma_{ji}(\lambda) = \beta_{ij}(\lambda)$ since the 2-morphism on the right and left is non-zero.

When $-\langle i, \lambda + \alpha_j \rangle + 1 \leq 0$ closing off the left side of (6.6) with $\langle i, \lambda + \alpha_j \rangle - 1$ dots and simplifying the left-hand side using the KLR relation (2.10) implies $\beta_{ij}(\lambda) = t_{ij}$ for all $i, j \in I$ and λ satisfying $1 + d_{ij} \leq \langle i, \lambda \rangle$. Similarly, (6.9) implies $\beta_{ij}(\lambda) = t_{ij}$ for all $i, j \in I$ and λ satisfying $\langle i, \lambda \rangle \leq 1$.

For $1 < \langle i, \lambda \rangle \leq d_{ij}$ glue equation (6.6) into the diagram



where $\langle i, \lambda \rangle - d_{ij} \leq m \leq \langle i, \lambda \rangle - 1$. Simplifying using (2.14) and (6.8) above completes the proof. \square

Remark. One can obtain bubble sliding relations by generalizing the arguments in the proof above. Closing off the left side of (6.6) with $\langle i, \lambda + \alpha_j \rangle - 1 + m$ dots, where $m \geq \max(-\langle i, \lambda + \alpha_j \rangle + 1, 0)$, then simplifying the left-hand side using the KLR relation (2.10) implies

$$(6.8) \quad \beta_{ij}(\lambda) \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom} \\ \text{dots } (\langle i, \lambda + \alpha_j \rangle - 1) + m \end{array} \right] \lambda = t_{ij} \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom} \\ \text{dots } (\langle i, \lambda \rangle - 1) + m \end{array} \right] \lambda + t_{ji} \left[\begin{array}{c} \text{bubble with } j \text{ top, } i \text{ bottom} \\ \text{dots } (\langle i, \lambda \rangle - 1) + m - d_{ij} \end{array} \right] \lambda + \sum_{p,q} s_{ji}^{pq} \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom} \\ \text{dots } (\langle i, \lambda \rangle - 1) + m + q - d_{ij} \end{array} \right] \lambda$$

Interchanging the labels of the strands in (6.5) and capping off the right strand with $-\langle i, \lambda - \alpha_i \rangle - 1 + m = -\langle i, \lambda \rangle + 1 + m$ dots, where $m \geq \max(\langle i, \lambda \rangle - 1, 0)$, implies

$$(6.9) \quad \beta_{ij}(\lambda) \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom} \\ \text{dots } -\langle i, \lambda - \alpha_i \rangle - 1 + m \end{array} \right] \lambda - \alpha_i = t_{ij} \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom} \\ \text{dots } (-\langle i, \lambda' \rangle - 1) + m - d_{ij} \end{array} \right] \lambda - \alpha_i + t_{ji} \left[\begin{array}{c} \text{bubble with } j \text{ top, } i \text{ bottom} \\ \text{dots } (-\langle i, \lambda' \rangle - 1) + m - d_{ij} \end{array} \right] \lambda - \alpha_i + \sum_{p,q} s_{ij}^{qp} \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom} \\ \text{dots } (-\langle i, \lambda' \rangle - 1) + m + q - d_{ij} \end{array} \right] \lambda - \alpha_i$$

with $\lambda' = \lambda - \alpha_i + \alpha_j$.

6.3. Q-Cyclicity. By uniqueness of the space of 2-morphisms in the appropriate degree we have

$$(6.10) \quad \left[\begin{array}{c} \text{diagram with } j \text{ top, } i \text{ bottom, } \lambda \text{ label} \end{array} \right] = b_{ij}(\lambda) \left[\begin{array}{c} \text{diagram with } i \text{ top, } j \text{ bottom, } \lambda \text{ label} \end{array} \right]$$

where $b_{ij}(\lambda) \in \mathbb{k}^\times$.

Glue a downward crossing defined analogously as in equation (4.4) to the bottom of (6.10) and simplify using biadjointness to get

$$(6.11) \quad \left[\begin{array}{c} \text{diagram with } j \text{ top, } i \text{ bottom, } \lambda \text{ label, and a downward crossing at the bottom} \end{array} \right] = b_{ij}(\lambda) \left[\begin{array}{c} \text{diagram with } i \text{ top, } j \text{ bottom, } \lambda \text{ label, and a downward crossing at the bottom} \end{array} \right]$$

If $\langle i, \lambda \rangle \leq -1$ then cap off the right most strands in both diagrams with $m = -\langle i, \lambda \rangle - 1$ dots, simplify using cyclicity for dots and using (6.5) to get

$$(6.12) \quad \left[\begin{array}{c} \text{diagram with } j \text{ top, } i \text{ bottom, } \lambda + \alpha_j \text{ label, and } m \text{ dots on the right} \end{array} \right] = b_{ij}(\lambda) t_{ji} \left[\begin{array}{c} \text{bubble with } i \text{ top, } j \text{ bottom, } m \text{ dots} \end{array} \right] \lambda + \alpha_j$$

Since $m = -\langle i, \lambda \rangle - 1$ and degree zero bubbles are multiplication by 1, simplifying the left-hand side using the KLR relation (2.12) implies $b_{ij}(\lambda) = t_{ij} t_{ji}^{-1}$ for $\langle i, \lambda \rangle \leq -1$.

If $\langle i, \lambda \rangle \geq -1$ then glue a downward crossing onto the top of (6.10) and close off the left most strand with $m = \langle i, \lambda + \alpha_i - \alpha_j \rangle - 1$ dots. Note that since $\langle i, \alpha_j \rangle \leq 0$ we have $m \geq 0$. Then arguing as above shows that $b_{ij}(\lambda) = t_{ij}t_{ji}^{-1}$. Thus, in all cases we have $b_{ij}(\lambda) = t_{ij}t_{ji}^{-1}$ for all $i, j \in I$ with $i \neq j$.

7. APPLICATIONS

7.1. Cohomology of iterated flag varieties. Extending results from [9, 1] an action of the 2-category $\mathcal{U}(\mathfrak{sl}_2)$ was constructed on categories of modules of cohomology rings [17] (as well as equivariant cohomology rings [18]) of partial flag varieties. This action was generalized to an action of $\mathcal{U}(\mathfrak{sl}_n)$ in [13] where it was used to prove nondegeneracy of the 2-category $\mathcal{U}(\mathfrak{sl}_n)$.

For these 2-representations on partial flag varieties construction of an action of the KLR algebras was fairly straightforward. However, verification of the extended \mathfrak{sl}_2 relations was rather complicated and involved comparing by hand certain bimodule maps. Our result 1.1 provides an immediate simplification since one only needs to check the $[E, F]$ commutation relation for \mathfrak{sl}_2 and then all the extended \mathfrak{sl}_2 relations follow formally.

7.2. Derived categories of coherent sheaves.

7.2.1. Cotangent bundles to Grassmannians. In [6] the idea of a “geometric categorical \mathfrak{sl}_2 action” was introduced. In [7] we showed that such an action induces a strong 2-representation of \mathfrak{sl}_2 , like the one in section 1.2. In [6] and [5] a geometric categorical \mathfrak{sl}_2 action was constructed on derived categories of coherent sheaves on cotangent bundles of Grassmannians $T^*\mathbb{G}(k, N)$.

If we put all these results together, we obtain a strong 2-representation of \mathfrak{sl}_2 where:

- the objects $\lambda \in \mathbb{Z}$ are $DCoh(T^*\mathbb{G}(k, N))$ where $N \in \mathbb{Z}^{\geq 0}$ is fixed, $0 \leq k \leq N$, $\lambda = N - 2k$ and $DCoh(X)$ denotes the bounded derived category of coherent sheaves on X .
- the 1-morphisms

$$\begin{aligned} E^{(r)} &: DCoh(T^*\mathbb{G}(k+r, N)) \rightarrow DCoh(T^*\mathbb{G}(k, N)) \text{ and} \\ F^{(r)} &: DCoh(T^*\mathbb{G}(k, N)) \rightarrow DCoh(T^*\mathbb{G}(k+r, N)) \end{aligned}$$

are induced by certain correspondences

$$T^*\mathbb{G}(k+r, N) \leftarrow W^r(k, N) \rightarrow T^*\mathbb{G}(k, N)$$

by pulling back, tensoring with a line bundle and pushing forward (Fourier-Mukai transforms).

- the action of the KLR algebra (a.k.a. the NilHecke algebra in this case) on the Es is given by the formal construction in [7].

Applying theorem 1.1 we find that:

Theorem 7.1. *The geometric categorical \mathfrak{sl}_2 action on $\bigoplus_{k=0}^N DCoh(T^*\mathbb{G}(k, N))$ defined in [6] and [5] induces a 2-representation $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ on \mathcal{K} where:*

- the objects of \mathcal{K} are the derived categories of coherent sheaves $DCoh(T^*\mathbb{G}(k, N))$,
- the 1-morphisms of \mathcal{K} are the Fourier-Mukai kernels in $DCoh(T^*\mathbb{G}(k+r, N) \times T^*\mathbb{G}(k, N))$,
- the 2-morphisms of \mathcal{K} are morphisms between these Fourier-Mukai kernels.

7.2.2. Nakajima quiver varieties. In [8] one extends the notion from [7] to define a “geometric categorical \mathfrak{g} action” where \mathfrak{g} is an arbitrary simply laced Kac-Moody Lie algebra. One also constructs such a geometric categorical \mathfrak{g} action on the derived categories of coherent sheaves on Nakajima quiver varieties (this generalizes the action mentioned above on cotangent bundles of Grassmannians).

Although the analogue of the result from [7] has not been established we believe it should be true, at least when the Dynkin diagram of \mathfrak{g} is a tree. More precisely,

Conjecture 7.1. If the Dynkin diagram of \mathfrak{g} is simply laced and contains no loops then a geometric categorical \mathfrak{g} action induces a strong 2-representation of \mathfrak{g} .

The difficult part in proving the conjecture above is showing that the KLR algebra acts on \mathbf{Es} . Once we know this the conjecture follows and then Theorem 1.1 implies that there is a 2-representation of $\dot{\mathcal{U}}(\mathfrak{g})$ on derived categories of coherent sheaves on Nakajima quiver varieties of type \mathfrak{g} (just like theorem 7.1 when $\mathfrak{g} = \mathfrak{sl}_2$).

7.3. Cyclotomic quotients. For $\Lambda = \sum_{i \in I} \lambda_i \Lambda_i \in X^+$ consider the two-sided ideal \mathcal{J}^Λ of $R = R_Q$ generated by elements

$$\begin{array}{c} \uparrow \\ \lambda_{i_1} \bullet \\ \downarrow \\ i_1 \end{array} \quad \begin{array}{c} \uparrow \\ \vdots \\ \downarrow \\ i_2 \end{array} \quad \cdots \quad \begin{array}{c} \uparrow \\ \vdots \\ \downarrow \\ i_m \end{array}$$

over all $(i_1, i_2, \dots, i_m) \in I^m$ and $m \geq 0$. Define the cyclotomic quotient of the KLR algebra as the quotient

$$(7.1) \quad R^\Lambda := R / \mathcal{J}^\Lambda.$$

The cyclotomic quotient conjecture from [12, 14] states that the category $\text{Proj}(R^\Lambda)$ of finitely generated projective R^Λ -modules categorifies the irreducible highest weight representation of $U_q(\mathfrak{g})$ with dominant integral weight Λ . This implies there is an isomorphism of $U_q(\mathfrak{g})$ -modules

$$V(\Lambda)_{\mathbb{Z}} \cong [\text{Proj}(R^\Lambda)],$$

where $V(\Lambda)_{\mathbb{Z}}$ is the integral version of the irreducible highest weight representation for $\dot{\mathcal{U}}(\mathfrak{g})$ associated to Λ and $[\text{Proj}(R^\Lambda)]$ is the Grothendieck ring of $\text{Proj}(R^\Lambda)$.

Parts of this conjecture have been proven in [3, 15, 2, 20, 16, 11]. Recently Kang and Kashiwara [10] prove this conjecture by showing that i -induction and i -restriction functors induce a functorial action of $U_q(\mathfrak{g})$ on the categories $\text{Proj}(R^\Lambda)$. They also define natural transformations between these functors giving an action of the KLR algebra [10, Section 6].

The category $\text{Proj}(R^\Lambda)$ decomposes as a direct sum of categories $\text{Proj}(R^\Lambda(\alpha))$ for $\alpha = (i_1, i_2, \dots, i_m) \in I^m$ lifting the weight spaces of the irreducible representation $V(\Lambda)$.

Conjecture 7.2. For any α corresponding to a nonzero weight space of $V(\Lambda)$ the center of the ring $R^\Lambda(\alpha)$ is zero dimensional in negative degrees and one dimensional in degree zero.

Corollary 7.2. Let R^Λ denote the cyclotomic quotient of the KLR algebra associated to a symmetrizable Kac-Moody algebra \mathfrak{g} for any choice of scalars Q . Then, assuming Conjecture 7.2, the 2-categories $\text{Proj}(R^\Lambda)$ form a 2-representation of $\mathcal{U}_Q(\mathfrak{g})$.

Remark. A statement similar to Corollary 7.2 appears in [23, Theorem 1.7 (version 6)]. However, the 2-category used there is not quite $\mathcal{U}_Q(\mathfrak{g})$ whenever the coefficient $t_{ij} \neq 1$.

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